In fact we can again be more precise for the 5-point matrix:

__Lemma__ the 5-point matrix A has eigenvectors $v^{r,s}$, for $r,s=1,\dots,n$ with entries

$$v^{rs}_{jk} = \sin\frac{jr\pi}{n+1}\,\sin\frac{ks\pi}{n+1}$$

$j,k=1,\dots,n$ and corresponding eigenvalues

$$\lambda^{rs} = \frac{1}{h^2}\left[4-2\cos\frac{r\pi}{n+1}-2\cos\frac{s\pi}{n+1}\right]$$

__Proof__ by direct calculation as in 1-D case (on problem sheet)

Note $\lambda_{max} = \frac{1}{h^2}\left[4-4\cos\frac{n\pi}{n+1}\right]$

and

$$\cos\frac{n\pi}{n+1} = \cos\left(\pi-\frac{\pi}{n+1}\right) = \cos\pi\cos\frac{\pi}{n+1} + \underbrace{\sin\pi\sin\frac{\pi}{n+1}}_{0}$$

$$= -1\left(1-\frac{\pi^2}{2(n+1)^2}+O(n^{-4})\right)$$

$$= -1+\frac{\pi^2}{2(n+1)^2}+O(h^4)$$

as $(n+1)h=1$

so $\lambda_{max} = 8h^{-2}-2\pi^2+O(h^2)$

By similar analysis

$$\lambda_{min} = \frac{1}{h^2}\left[4-4\cos\frac{\pi}{n+1}\right]$$

$$= \frac{1}{h^2}\left[4-4\left(1-\frac{\pi^2}{2(n+1)^2}+O(h^4)\right)\right]$$

$$= 2\pi^2-O(h^2)$$

This knowledge of the eigenvalues gives an immediate error bound as in 1-D case:

If the finite difference equations are

$$L_h u_{jk} = f(x_j,y_k)$$

and by definition

$$L_h u(x_j,y_k) = f(x_j,y_k)+\tau_{jk}$$

local truncation error

then with $e_{jk} = u(x_j,y_k)-u_{jk}$

subtract $\Rightarrow$

$$L_h e_{jk} = \tau_{jk} \qquad j,k=1,\dots,n$$

$$0 = e_{0k} = e_{n+1\,k} = e_{j0} = e_{j\,n+1}$$

is the same as $Ae=\tau$

$e=\{e_{jk}\}$, $\tau=\{\tau_{jk}\}$

and so $e = A^{-1}\tau$ from which follows

$$\|e\| \le \|A^{-1}\|\,\|\tau\| = \frac{1}{\lambda_{min}(A)}\|\tau\|$$

so $\|e\| \le \frac{1}{2\pi^2}\|\tau\|$.

We need to know the local truncation error: as in 1-D

$$\frac{-u(x_{j+1},y_k)+2u(x_j,y_k)-u(x_{j-1},y_k)}{h^2} = -\frac{\partial^2 u}{\partial x^2}(x_j,y_k)+O(h^2)$$

and similarly

$$\frac{-u(x_j,y_{k+1})+2u(x_j,y_k)-u(x_j,y_{k-1})}{h^2} = -\frac{\partial^2 u}{\partial y^2}(x_j,y_k)+O(h^2)$$

so $\tau_{jk} = O(h^2)$ so $\|e\| \le \frac{1}{2\pi^2}\|\tau\|$ ← entries $O(h^2)$

where $\|u\| = \left(\sum_{j=1}^{n}\sum_{k=1}^{n}u_{jk}^2\right)^{1/2}$ is the usual Euclidean length of a vector

Alternatively a pointwise result (rather than one based on norms) could be achieved by the maximum/minimum principle.

__Lemma__ (Maximum Principle)

If $4\phi_{jk}-\phi_{j+1\,k}-\phi_{j-1\,k}-\phi_{j\,k+1}-\phi_{j\,k-1} \le 0$ (*)

$j,k=1,\dots,n$

then $\phi_{jk} \le \max\left\{0,\ \max_j\{\phi_{j0},\phi_{j\,n+1}\},\ \max_k\{\phi_{0k},\phi_{n+1\,k}\}\right\}$

i.e. the maximum is taken on the boundary or is zero.

__Proof__ Assume not and ϕ_{jk} is max for some j,k with $1\le j\le n$, $1\le k\le n$

then (*) $\Rightarrow$ $\phi_{jk} = \phi_{j+1\,k} = \phi_{j-1\,k} = \phi_{j\,k+1} = \phi_{j\,k-1}$

and continue until 'hit' the boundary

__Corollary__ (Minimum Principle)

If $4\phi_{jk}-\phi_{j+1\,k}-\phi_{j-1\,k}-\phi_{j\,k+1}-\phi_{j\,k-1} \ge 0$

$j,k=1,\dots,n$

then min value of ϕ_{jk} is taken on the boundary or is zero.

Consider the non-negative mesh function

$$\psi_{jk} = (x_j-\tfrac{1}{2})^2+(y_k-\tfrac{1}{2})^2$$

and let $\phi_{jk} = e_{jk}+\frac{1}{4}\tau\psi_{jk}$

where $\tau = \max_{j,k}|\tau_{jk}|$

and $e_{jk} = u(x_j,y_k)-u_{jk}$ is the error.

If now L_h is the 5-point finite difference defined by

$$L_h u_{jk} = \frac{1}{h^2}\left(4u_{jk}-u_{j+1\,k}-u_{j-1\,k}-u_{j\,k+1}-u_{j\,k-1}\right)$$

then

$$L_h\phi_{jk} = \underbrace{L_h e_{jk}}_{=\tau_{jk}} + \frac{1}{4}\tau\underbrace{L_h\psi_{jk}}_{-4}$$

(see problem sheet)

so $L_h\phi_{jk} = \tau_{jk}-\tau \le 0$

hence ϕ_{jk} takes its maximum on the boundary by Max. Princ.

i.e. $e_{jk}+\frac{1}{4}\tau\psi_{jk} \le \max_{\text{on bdry}}\{e_{jk}+\frac{1}{4}\tau\psi_{jk}\}$ ($e_{jk}=0$ on bdry)

$$\le \frac{1}{8}\tau$$

as $\psi_{jk} \le \frac{1}{2}$ on $\Omega = [0,1]\times[0,1]$

$\Rightarrow$ $e_{jk} = u(x_j,y_k)-u_{jk} \le \frac{1}{8}\tau$ as ψ_{jk} is non-negative

Similarly applying the Minimum Principle $\Rightarrow$

$$-\frac{1}{8}\tau \le u(x_j,y_k)-u_{jk} \le \frac{1}{8}\tau,$$

$$\tau = O(h^2)$$