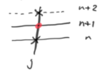
$-\nabla^2 u = f$ in Ω, u given on $\partial\Omega$ is the Dirichlet problem for the Poisson Equation: u given on $\partial\Omega$ are Dirichlet boundary conditions.

Neumann boundary conditions would be $\frac{\partial u}{\partial n} = \nabla u \cdot n$ given on the boundary where n is the outwards pointing normal to Ω on $\partial\Omega$: gives the Neumann problem for the Poisson Equation.

Approximation: as in 1-D

e.g. $\frac{\partial u}{\partial n}$ $\left(= \frac{\partial u}{\partial y}\right.$ on top and $-\frac{\partial u}{\partial y}$ on bottom of unit square $)$

so $\frac{\partial u}{\partial y} = \ell(x)$ on top say



approximated by $\frac{u_{j\,n+2} - u_{j\,n}}{2h} = \ell(x_j)$ each j

and the 'fictitious' value $u_{j\,n+2}$ can be eliminated from

$$4u_{j\,n+1} - u_{j+1\,n+1} - u_{j-1\,n+1} - u_{j\,n+2} - u_{j\,n} = f(x_j, y_{n+1})$$

giving n extra equations

$$4u_{j\,n+1} - u_{j+1\,n+1} - u_{j-1\,n+1} - 2u_{j\,n} = f(x_j, y_{n+1}) - 2h\,\ell(x_j) \qquad j = 1, \dots, n$$

as well as the usual n^2 equations for $k = 1, \dots, n$ and $j = 1, \dots, n$.

Finite differences can be applied on domains Ω which are not square (!) (see problem sheet), but for general curved boundaries even though interpolation can be used, in general one should consider finite element approximation instead.

Rotated 5-point formula

In 2D we have more flexibility and other schemes are possible: By Taylor's Theorem in 2 variables

$$u(x_{j\pm 1}, y_{k\pm 1}) = u(x_j \pm h, y_k + h)$$

$$= u \pm h u_x \pm h u_y + \frac{h^2}{2} u_{xx} \pm h^2 u_{xy} + \frac{h^2}{2} u_{yy}$$

$$\pm \frac{h^3}{6} u_{xxx} \pm \frac{3h^3}{6} u_{xxy} \pm \frac{3h^3}{6} u_{xyy} \pm \frac{h^3}{6} u_{yyy} + O(h^4)$$

u, u_x etc all evaluated at (x_j, y_k).

, so

$$u(x_{j+1}, y_{k+1}) + u(x_{j-1}, y_{k+1})$$
$$= 2u + 2hu_y + 2\frac{h^2}{2} u_{xx} + 2\frac{h^2}{2} u_{yy} + 2\frac{h^3}{6} u_{xxy} + 2\frac{h^3}{6} u_{yyy} + O(h^4)$$

and

$$u(x_{j+1}, y_{k-1}) + u(x_{j-1}, y_{k-1})$$
$$= 2u - 2hu_y + 2\frac{h^2}{2} u_{xx} + 2\frac{h^2}{2} u_{yy} - 2\frac{h^3}{2} u_{xxy} - 2\frac{h^3}{6} u_{yyy} + O(h^4)$$

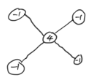
Add

$$\overline{4u \quad + 2h^2 u_{xx} + 2h^2 u_{yy} \quad + O(h^4)}$$

So

$$\frac{4u_{jk} - u_{j+1\,k+1} - u_{j-1\,k+1} - u_{j+1\,k-1} - u_{j-1\,k-1}}{2h^2} \sim -u_{xx} - u_{yy} \text{ with } \tau_{jk} = O(h^2).$$

approximates



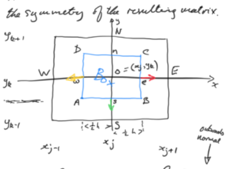
For obvious reasons this is called the rotated 5-point formula / stencil

We can also consider variable coefficient problems

$$f = \nabla \cdot \left(p(x,y)\nabla u\right) = \frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(p(x,y)\frac{\partial u}{\partial y}\right)$$

(Note sign: $p = -1$ would give $-\nabla^2 u$)

As in 1-D it is convenient to ensure the symmetry of the resulting matrix.



$$\int_{\text{Box}} f = \int_{\text{Box}} \nabla \cdot (p\nabla u) = \int_{\partial\,\text{Box}} p\nabla u \cdot n$$

Divergence Theorem

outwards normal

$$= -\int_A^B p\frac{\partial u}{\partial y} + \int_B^C p\frac{\partial u}{\partial x} + \int_C^D p\frac{\partial u}{\partial y} - \int_D^A p\frac{\partial u}{\partial x}$$

so approximate by

$$h\,p(\underbrace{x_j, y_{k-\frac{1}{2}}}_{s})\frac{u_S - u_0}{h} + h\,p(\underbrace{x_{j+\frac{1}{2}}, y_k}_{e})\frac{u_E - u_0}{h}$$
$$+ h\,p(x_j, y_{k+\frac{1}{2}})\frac{u_N - u_0}{h} + h\,p(x_{j-\frac{1}{2}}, y_k)\frac{u_W - u_0}{h} = h^2 f(x_j, y_k)$$

$u_0 = u_{jk}$, $u_E = u_{j+1\,k}$, $u_N = u_{j\,k+1}$ etc

At boundaries need consider boxes like

Proposition the resulting matrix is symmetric.

Proof exercise: see problem sheet

Remark:

This idea of integrating the PDE over a small volume, using the Divergence Theorem and approximating line integrals leads to so-called Finite Volume methods which