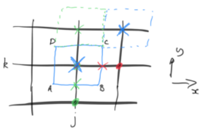
Finite Volume Methods

for $\nabla\cdot(u,v) = f$



$$\iint_{box} \nabla\cdot(u,v) = \iint_{box} f$$

$$\parallel$$

$$\int_{\partial box} (u,v)\cdot \underline{n}\, dS$$

~~approximated by~~ equal to

$$\int_A^B -v + \int_B^C u + \int_C^D v + \int_D^A -u = \iint_{box} f$$

approximated by

$$-h\left(\frac{v_{j\,k-1}+v_{jk}}{2}\right) + h\left(\frac{u_{j+1\,k}+u_{jk}}{2}\right)$$

$$+h\left(\frac{v_{j\,k+1}+v_{jk}}{2}\right) - h\left(\frac{u_{j-1\,k}+u_{jk}}{2}\right)$$

$$= h^2 f_{jk}$$

Numerical Methods for Hyperbolic Conservation Laws

Conservation Laws are of the form

$$\frac{\partial u}{\partial t} + \nabla\cdot \underline{f}(u) = 0$$

where $\underline{f}$ is the flux.

So called because in any region Ω with boundary $\partial\Omega$, by integration we get

$$\frac{d}{dt}\iiint_\Omega u\, dV + \underbrace{\iiint_\Omega \nabla\cdot \underline{f}(u)\, dV}_{\iint_{\partial\Omega} \underline{f}(u)\cdot\underline{n}\, dS} = 0$$

so: rate of change of u in Ω

= flux of u into Ω across $\partial\Omega$ (i.e. in the direction $-\underline{n}$)

In 1 spatial dimension

$$u_t + f(u)_x = 0$$

$$\Rightarrow \frac{d}{dt}\int_a^b u\,dx = -f(b) + f(a)$$

so the total amount of u is conserved

If $a(u) = f'(u)$ then $u_t + a(u)u_x = 0$

and we have that for the solution $u(x,t)$

$$\frac{d}{dt}\left(u(x(t),t)\right) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

Chain Rule

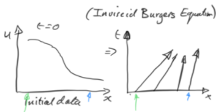
$$= \left[\frac{dx}{dt} - a(u)\right]\frac{\partial u}{\partial x}$$

$= 0$ along curves $\frac{dx}{dt} = a(u)$

i.e. u is constant on $\frac{dx}{dt} = a(u)$: characteristic curves and $a(u)$ constant on characteristic curves $\Rightarrow$ characteristic curves are straight lines determined by initial data

Examples (1) $u_t + u u_x = 0$

(Inviscid Burgers Equation)



characteristics: can cross

$\Rightarrow$ solution discontinuities or 'shocks'.

(2) $u_t + a u_x = 0$, $a\in\mathbb{R}$ constant

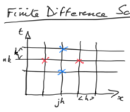
1st order Wave Equation: characteristics are lines of slope $1/a$ in x-t space

This equation is also called the advection equation.

We will initially concentrate on this linear wave equation and assume $a > 0$ for definiteness

($\Rightarrow$ right moving wave solution)

Finite Difference Schemes



$u_j^n \simeq u(jh, nk)$

(sometimes $h = \Delta x$, $k = \Delta t$)

Simplest (A)

$$\frac{u_j^{n+1} - u_j^{n-1}}{2k} + a\,\frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

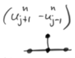
or $u_j^{n+1} = u_j^{n-1} - \nu\left(u_{j+1}^n - u_{j-1}^n\right)$, $\nu = \frac{ak}{h}$

'Leapfrog' scheme

(a 3 level scheme since 3 time level involved)

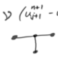
or (B)

$$u_j^{n+1} = u_j^n - \tfrac{1}{2}\nu\left(u_{j+1}^n - u_{j-1}^n\right)$$



turns out to be unstable, so not useful

but $u_j^{n+1} = u_j^n - \frac{1}{2}\nu\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right)$



the "Backward Euler" scheme is implicit

(C) Lax-Friedrichs scheme

$$u_j^{n+1} = \tfrac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \tfrac{1}{2}\nu\left(u_{j+1}^n - u_{j-1}^n\right)$$