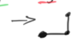
$$u_t + a u_x = 0 \quad , a \in \mathbb{R}^+$$

or using 1-sided differences

(D) 1^{st} order upwind scheme

$$u_j^{n+1} = u_j^n - \nu\left(u_j^n - u_{j-1}^n\right)$$

 (since $a > 0$)

Each scheme has a local truncation error defined as before e.g. for 1^{st} order upwind

$$\tau_j^n = \frac{u(x_j, t^n + k) - u(x_j, t^n)}{k} + a\,\frac{u(x_j, t^n) - u(x_j - h, t^n)}{h}$$

$$\underset{\substack{\uparrow \\ \text{Taylor} \\ \text{series}}}{=} u_t + \tfrac{1}{2} k u_{tt} + O(k^2) + a\left(u_x - \tfrac{1}{2} h u_{xx} + O(h^2)\right)$$

$$= \overbrace{u_t + a u_x}^{0} + O(k) + O(h)$$

hence $\tau_j^n \to 0$ as k & $h \to 0$

so 1^{st} order accurate in space and time

(Note u_t, u_{tt}, u_x etc all evaluated at (x_j, t^n))

For consistency we require $\tau_j^n \to 0$ as $h, k \to 0$ similarly to before

Scheme (A) is 2^{nd} order accurate in space and time

(B) is 2^{nd} order accurate in space, 1^{st} order accurate in time

(C) is 1^{st} order accurate in space and in time

To derive a more accurate 2 level scheme use Taylor Series and the PDE

$$u(jh, (n+1)k) = u + k u_t + \frac{k^2}{2} u_{tt} + \frac{k^3}{6} u_{ttt} + O(k^4)$$

where u, u_t etc evaluated at (jh, nk).

But $u_t = -a u_x \Rightarrow u_{tt} = -a u_{xt}$

$$= -a(u_t)_x = a^2 u_{xx}$$

So

$$u(jh, (n+1)k) = u - ak u_x + \frac{a^2 k^2}{2} u_{xx} + O(k^3)$$

and approximating with 2^{nd} order spatial finite differences leads to the **Lax-Wendroff** scheme

$$u_j^{n+1} = u_j^n - \tfrac{1}{2}\nu\left(u_{j+1}^n - u_{j-1}^n\right) + \tfrac{1}{2}\nu^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

which has $\tau_j^n = \frac{1}{6}\left(k^2 u_{ttt} - ah^2 u_{xxx}\right)$

so 2^{nd} order accurate in space and time.

Writing U^n for the vector of approximations u_j^n at time $t^n = nk$. All of these 2 level schemes can be expressed as

$$u_j^{n+1} = \mathcal{H}_k(U^n; j)$$

e.g. For Lax-Friedrichs we have

$$\mathcal{H}_k(U^n; j) = \tfrac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \tfrac{1}{2}\frac{ak}{h}\left(u_{j+1}^n - u_{j-1}^n\right)$$

A key aspect for all such schemes is **stability**

Definition a scheme $u_j^{n+1} = \mathcal{H}_k(U^n; j)$ is stable if for each T, $\exists$ a constant C and $K > 0$ such that

$$\|(\mathcal{H}_k)^n\| \le C \text{ for all } nk \le T, \; k < K$$

This is certainly satisfied if $\|\mathcal{H}_k\| \le 1$ since $\|(\mathcal{H}_k)^n\| \le \|\mathcal{H}_k\|^n \le 1$ all n, k

but more generally some growth is allowed e.g. if $\|\mathcal{H}_k\| \le 1 + \alpha k$, some $\alpha \in \mathbb{R}^+$

$\|\mathcal{H}_k^n\| \le (1 + \alpha k)^n \le e^{\alpha k n} \le e^{\alpha T}$ all k with $nk \le T$

For example : Stability of Lax-Friedrichs for $u_t + a u_x = 0$: here $\mathcal{H}_k$ is linear so $u_j^{n+1} = \mathcal{H}_k U^n$:

i.e. $u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{ak}{2h}\left(u_{j+1}^n - u_{j-1}^n\right)$

$$= \tfrac{1}{2}\left(1 - \tfrac{ak}{h}\right) u_{j+1}^n + \tfrac{1}{2}\left(1 + \tfrac{ak}{h}\right) u_{j-1}^n$$

and so using

$$\underbrace{\|U^{n+1}\| = h \sum_j |u_j^{n+1}|}_{\text{mimics } \int_a^b |u^{n+1}|\,dx} \le \frac{h}{2}\left[\left|\sum_j \left(1 - \tfrac{ak}{h}\right) u_{j+1}^n\right| + \left|\sum_j \left(1 + \tfrac{ak}{h}\right) u_{j-1}^n\right|\right]$$

$$\le \frac{h}{2}\left[\left(1 - \tfrac{ak}{h}\right)\sum_j |u_{j+1}^n| + \left(1 + \tfrac{ak}{h}\right)\sum_j |u_{j-1}^n|\right]$$

provided $1 - \frac{ak}{h} \ge 0$, $1 + \frac{ak}{h} \ge 0$

$$= \tfrac{1}{2}\left[\left(1 - \tfrac{ak}{h}\right)\|U^n\| + \left(1 + \tfrac{ak}{h}\right)\|U^n\|\right]$$

$$= \|U^n\|$$

Thus $\|U^{n+1}\| \le \|U^n\|$ when $\left|\frac{ak}{h}\right| \le 1$

i.e. $\|\mathcal{H}_k\| \le 1$ for the Lax-Friedrichs scheme provided $\left|\frac{ak}{h}\right| \le 1$