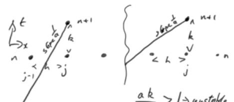
A simple necessary condition for stability is the

CFL (Courant, Freidrichs, Lewy) condition

which requires that the Domain of Dependence of the finite difference scheme must contain the Domain of Dependence of the PDE

For $u_t + au_x = 0$ this means that the characteristic through (x_j, t^{n+1}) must lie within the finite difference stencil

i.e. $\left|\frac{ak}{h}\right| \leq 1$ for any stencil



CFL satisfied

$\frac{ak}{h} > 1 \Rightarrow$ unstable

Together with the analysis above, we conclude the Lax-Freidrichs scheme is stable if and only if $\left|\frac{ak}{h}\right| \leq 1$

Note: CFL applies for both $a > 0$ and $a < 0$ $\Rightarrow$

(leftwards moving waves)

Stability is important because

Lax Equivalence Theorem

Given a well-posed linear Initial Value Problem and a consistent constant coefficient finite difference approximation to it, then stability is necessary and sufficient for convergence

Recall:
- consistent $\Leftrightarrow$ $\tau \to 0$ as $h, k \to 0$
- convergence $\Leftrightarrow$ $\| u(jh, nk) - U_j^n \| \to 0$ as $h, k \to 0$

Linear Systems of Hyperbolic PDEs.

$$\underline{u}_t + A\,\underline{u}_x = 0 \quad , \quad A \in \mathbb{R}^{p \times p}$$

a constant matrix

1) Centred schemes (stencil does not depend on the sign of a for $u_t + au_x = 0$)

e.g. Lax-Wendroff

$$\underline{u}_j^{n+1} = \underline{u}_j^n - \frac{1}{2}\frac{k}{h}A\left(\underline{u}_{j+1}^n - \underline{u}_{j-1}^n\right) + \frac{1}{2}\frac{k^2}{h^2}A^2\left(\underline{u}_j^{n+1} - 2\underline{u}_j^n + \underline{u}_{j-1}^n\right)$$

i.e. nothing much changes!

2) Upwind schemes (stencil depends on $\mathrm{Sign}(a)$)

e.g. 1st order upwind: must decompose into Riemann Invariants

i.e. diagonalise $A = X \Lambda X^{-1}$

i.e. $AX = X\Lambda$

or equivalently $A\underline{x}_i = \lambda_i \underline{x}_i$ $i = 1, \dots, p$

where $\underline{x}_i$ is the i^{th} column of X $i = 1, \dots, p$

and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_p)$.

Then $\underline{u}_t + X\Lambda X^{-1}\underline{u}_x = 0$

$$\Leftrightarrow \quad (X^{-1}\underline{u})_t + \Lambda (X^{-1}\underline{u})_x = 0 \leftarrow$$

so writing $\underline{v} = X^{-1}\underline{u}$, the components v_i of $\underline{v}$ are the Riemann Invariants which satisfy the scalar PDEs

$$(v_i)_t + \lambda_i (v_i)_x = 0 \quad \leftarrow$$

and now the upwind scheme depends on the sign (λ_i)

$\lambda_i > 0$ $\lambda_i < 0$

e.g. Wave Equation $u_{tt} - a^2 u_{xx} = 0$

write $r = u_t$, $s = u_x$ so

$$s_t - r_x = 0$$

and wave eqn is $r_t - a^2 s_x = 0$

i.e. $$\begin{bmatrix} r \\ s \end{bmatrix}_t + \underbrace{\begin{bmatrix} 0 & -a^2 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} r \\ s \end{bmatrix}_x = 0$$

$\underline{u} = \begin{bmatrix} r \\ s \end{bmatrix}$ in notation above

A has eigenpairs $\lambda_1 = -a$, $\underline{x}_1 = \begin{bmatrix} a \\ 1 \end{bmatrix}$; $\lambda_2 = a$, $\underline{x}_2 = \begin{bmatrix} -a \\ 1 \end{bmatrix}$

so $(v_1)_t - a(v_1)_x = 0$, $(v_2)_t + a(v_2)_x = 0$

where $X = \begin{bmatrix} a & -a \\ 1 & 1 \end{bmatrix}$, $X^{-1} = \frac{1}{2a}\begin{bmatrix} 1 & a \\ -1 & a \end{bmatrix}$

$$v_1 = \frac{1}{2a} r + \frac{1}{2} s = \frac{1}{2a} u_t + \frac{1}{2} u_x$$

$$v_2 = -\frac{1}{2a} r + \frac{1}{2} s = \frac{1}{2a} u_t + \frac{1}{2} u_x$$

So 1st order upwind scheme is

$$(v_1)_j^{n+1} = (v_1)_j^n + a\frac{k}{h}\left[(v_1)_{j+1}^n - (v_1)_j^n\right] \quad (└)$$

$$(v_2)_j^{n+1} = (v_2)_j^n - a\frac{k}{h}\left[(v_2)_j^n - (v_2)_{j-1}^n\right] \quad (┘)$$

or since $\underline{v} = X^{-1}\underline{u}$, $\underline{u} = X\underline{v}$ i.e.

$$\underline{u} = v_1 \underline{x}_1 + v_2 \underline{x}_2$$

$$\underline{u}_j^{n+1} = (v_1)_j^{n+1}\underline{x}_1 + (v_2)_j^{n+1}\underline{x}_2$$

$$= \underline{u}_j^n + a\frac{k}{h}\left[(v_1)_{j+1}^n - (v_1)_j^n\right]\underline{x}_1 - a\frac{k}{h}\left[(v_2)_j^n - (v_2)_{j-1}^n\right]\underline{x}_2$$

so

$$\underline{u}_{j+1}^n = \underline{u}_j^n - \frac{k}{h}\left[X\begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix}X^{-1}\left(\underline{u}_{j+1}^n - \underline{u}_j^n\right)\right.$$