$$\underline{u}_t + A\underline{u}_x = 0$$

In general if $A = X \Lambda X^{-1}$

with $\Lambda = \Lambda^- + \Lambda^+$

$$\Lambda^- = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_q & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}, \qquad \Lambda^+ = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \lambda_{q+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_p \end{bmatrix}$$

$\lambda_1, \dots, \lambda_q < 0$, $\lambda_{q+1}, \dots, \lambda_p > 0$

then if we write $A^- = X \Lambda^- X^{-1}$

$$A^+ = X \Lambda^+ X^{-1}$$

the first order upwind scheme is

$$\underline{u}_j^{n+1} = \underline{u}_j^n - \frac{k}{h}\left[A^-\left(\underline{u}_{j+1}^n - \underline{u}_j^n\right) + A^+\left(\underline{u}_j^n - \underline{u}_{j-1}^n\right)\right]$$

which is sometimes called the Courant–Isaacson–Rees method

Clearly the eigenvalues take the place of the wavespeed, a, in the scalar equation, so the (necessary) CFL stability condition become

$$\max_j \left|\lambda_j \frac{k}{h}\right| \le 1$$

An unfortunate property of most schemes is that they are either

1) dissipative : wave amplitudes decay as time increases

2) dispersive : different frequency waves travel at different speeds

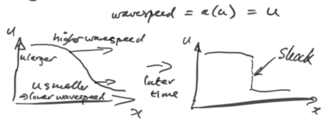
1) is often associated with leading local truncation error terms which involve the term u_{xx} associated with diffusion / decay of solutions

2) often leads to Gibbs-like phenomenon

These issues arise for linear and nonlinear wave problems. Associated only with nonlinear problems can also be the formation of shocks : discontinuities that form as a solution evolves.

e.g. Burgers Equation $\quad u_t + uu_x = 0$

$$\text{wavespeed} = a(u) = u$$



Perhaps surprisingly finite difference (and finite volume) methods remain the most important and widely used methods for such problems : we consider these now

Scalar nonlinear conservation laws

$$u_t + f(u)_x = 0$$

or $\quad u_t + a(u)\, u_x = 0 \;, \; a(u) = f'(u)$

can be approximated by locally freezing wavespeeds :

$$a_{j+\frac{1}{2}}^n = \begin{cases} \dfrac{f_{j+1}^n - f_j^n}{u_{j+1}^n - u_j^n} & , u_{j+1}^n \neq u_j^n \\ f'(u_j^n) & \end{cases} ,$$

$$f_j^n = f(u_j^n), \dots \text{etc}$$

The e.g. 1st order upwind scheme is for

$$\nu_{j+\frac{1}{2}}^n = a_{j+\frac{1}{2}}^n \frac{k}{h} \quad \text{etc}$$

For $j = 1, \dots, J$ (i.e. sweep from left to right update u_j^n either once, twice or not at all depending on local ν values):

$$u_j^{n+1} = u_j^n - \frac{k}{h} \Bigg\{$$