Conservative Schemes

If a finite difference scheme can be written in the form

$$u_j^{n+1} = u_j^n - \frac{k}{h}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right)$$

where $F_{j+\frac{1}{2}} = F\left(u_{j-p}^n, \dots, u_{j+q}^n\right)$ and if F is a "<u>numerical flux function</u>" consistent with the analytic flux $f(u)$ in the sense

$$F(u, u, \dots, u) = f(u)$$

then the scheme is said to be in "<u>conservation form</u>" and any scheme which can be written in this form is <u>a conservative scheme</u>.

This is because summing over any interval w.l.o.g. say $x_1, \dots, x_n$

$$\begin{aligned}\sum_{j=1}^{n} u_j^{n+1} = \sum_{j=1}^{n} u_j^n &- \frac{k}{h}\left[\cancel{F_{3/2}} - F_{1/2}\right] \\ &- \frac{k}{h}\left[\cancel{F_{5/2}} - \cancel{F_{3/2}}\right] \\ &\ \ \vdots \\ &- \frac{k}{h}\left[F_{n+\frac{1}{2}} - \cancel{F_{n-\frac{1}{2}}}\right] \\ = \sum_{j=1}^{n} u_j^n &+ \frac{k}{h}\left[F_{1/2} - F_{n+\frac{1}{2}}\right]\end{aligned}$$

because of telescopic cancellation

In other words

$$\frac{h\sum_{j=1}^{n} u_j^{n+1} - h\sum_{j=1}^{n} u_j^n}{k} = \frac{\|u^{n+1}\| - \|u^n\|}{k}$$

is only changed by the flux entering at the left hand of the interval and leaving at the right hand end

(With no boundary fluxes, $\sum_{j=1}^{n} u_j^n$ is conserved)

This mimics an important property of the underlying PDE problem.

All the 2-level schemes we have seen so far are conservative

e.g. Lax-Freidrichs scheme for $u_t + f_x = 0$

$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{k}{2h}\left[f(u_{j+1}^n) - f(u_{j-1}^n)\right]$$

can be written in conservation form with

$$\begin{aligned}F_{j+\frac{1}{2}} = F(u_j, u_{j+1}) = \frac{h}{2k}&\left(u_j - u_{j+1}\right) \\ &+ \frac{1}{2}\left[f(u_j) + f(u_{j+1})\right]\end{aligned}$$

which certainly satisfies $F(u, u) = f(u)$.

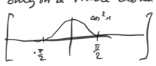
Since solutions to $u_t + f(u)_x = 0$ can involve shocks, we have to generalise the notion of a solution of the differential equation.

<u>Definition</u>

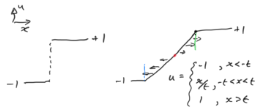
a bounded function u is called a <u>weak solution</u> of $u_t + f(u)_x = 0$ if

$$\int_{-\infty}^{\infty} \phi(x)\left(u_t + f(u)_x\right) dx = 0$$

for all sufficiently smooth functions ϕ with compact support (i.e. which are non-zero only on a finite closed interval.)



A regular (strong) solution is certainly a weak solution, but by weakening solutions we may loose uniqueness: e.g.



are both weak solutions of $u_t + uu_x = 0$

<u>Theorem</u> (Lax-Wendroff)

Given a conservative and consistent finite difference scheme for a conservation law, if the finite difference solution converges as $h \to 0$, $\frac{k}{h}$ fixed to a TV bounded function, then the limit function is a weak solution of the conservation law

This has the important consequence of 'shock capturing' since any shocks present must have the correct speed and location at least for small enough h

Mentioned here was the Total Variation (TV) defined in the non-discrete case as

$$TV(v) = \sup \sum_{j=1}^{N} |v(\xi_j) - v(\xi_{j-1})|$$

where the sup is over all subdivisions

$$-\infty < \xi_0 < \xi_1 < \dots < \xi_N < \infty$$

of $\mathbb{R}$.

In the discrete situation we more easily have

$$TV(u^n) = \sum_j |u_j^n - u_{j-1}^n|$$

hence new extrema $\Rightarrow TV(u^{n+1}) > TV(u^n)$

and since analytic and weak solutions of conservation laws are solutions with non-increasing TV, finite difference schemes that guarantee

$$TV(u^{n+1}) \le TV(u^n),$$