TVD Schemes

Write $\delta u^n_{j-\frac{1}{2}} = u^n_j - u^n_{j-1}$

Consider a scheme written as

$$u^{n+1}_j = u^n_j - C^n_{j-\frac{1}{2}}\,\delta u^n_{j-\frac{1}{2}} + D^n_{j+\frac{1}{2}}\,\delta u^n_{j+\frac{1}{2}}$$

where the coefficients C, D may depend on u^n

[e.g. Lax-Friedrichs for linear advection

$$u^{n+1}_j = u^n_j - \frac{1}{2}\left(1+\frac{ak}{h}\right)\delta u^n_{j-\frac{1}{2}} + \frac{1}{2}\left(1-\frac{ak}{h}\right)\delta u^n_{j+\frac{1}{2}}$$
]

There follows

$$\delta u^{n+1}_{j-\frac{1}{2}} = C^n_{j-\frac{3}{2}}\,\delta u^n_{j-\frac{3}{2}} + \left(1 - C^n_{j-\frac{1}{2}} - D^n_{j-\frac{1}{2}}\right)\delta u^n_{j-\frac{1}{2}} + D^n_{j+\frac{1}{2}}\,\delta u^n_{j+\frac{1}{2}}$$

so

$$\sum_j \left|\delta u^{n+1}_{j-\frac{1}{2}}\right| \le \sum_j \left|C^n_{j-\frac{3}{2}}\right|\left|\delta u^n_{j-\frac{3}{2}}\right| + \sum_j \left|1 - C^n_{j-\frac{1}{2}} - D^n_{j-\frac{1}{2}}\right|\left|\delta u^n_{j-\frac{1}{2}}\right| + \sum_j \left|D^n_{j+\frac{1}{2}}\right|\left|\delta u^n_{j+\frac{1}{2}}\right|$$

$$= \sum_j \left(\left|C^n_{j-\frac{1}{2}}\right| + \left|1 - C^n_{j-\frac{1}{2}} - D^n_{j-\frac{1}{2}}\right| + \left|D^n_{j-\frac{1}{2}}\right|\right)\left|\delta u^n_{j-\frac{1}{2}}\right|$$

so provided

$$\left.\begin{aligned} C^n_{j-\frac{1}{2}} &\ge 0 \\ 1 - C^n_{j-\frac{1}{2}} - D^n_{j-\frac{1}{2}} &\ge 0 \\ D^n_{j-\frac{1}{2}} &\ge 0 \end{aligned}\right\} (*)$$

we have $\sum_j \left|\delta u^{n+1}_{j-\frac{1}{2}}\right| \le \sum_j \left|\delta u^n_{j-\frac{1}{2}}\right|$

i.e. $TV(u^{n+1}) \le TV(u^n)$.

The conditions $(*)$ are thus sufficient for a scheme to be TVD

E.g. for 1st order wave equation, Lax-Friedrichs satisfies these conditions, but Lax-Wendroff does not (see problem sheet), however it can be modified by the inclusion of <u>flux-limiters</u>:

Let $\theta_j = \dfrac{u_j - u_{j-1}}{u_{j+1} - u_j}$ be the ratio of consecutive gradients

then writing Lax-Wendroff as ($\nu = \frac{ak}{h}$ as usual)

$$u^{n+1}_j = \underbrace{u^n_j - \nu\left(u^n_j - u^n_{j-1}\right)}_{1^{st}\text{ order upwind}} - \underbrace{\frac{1}{2}\nu(1-\nu)\left(u^n_{j+1} - 2u^n_j + u^n_{j-1}\right)}_{\substack{\text{anti-diffusive flux} \\ (\simeq -u_{xx} * \text{constant})}}$$

we can limit the anti-diffusive flux to satisfy the TVD criteria $(*)$:

$$u^{n+1}_j = u^n_j - \nu\left(u^n_j - u^n_{j-1}\right) - \frac{1}{2}\nu(1-\nu)\left[\phi_j\left(u^n_{j+1} - u^n_j\right) - \phi_{j-1}\left(u^n_j - u^n_{j-1}\right)\right]$$

where $\phi_j = \phi(\theta_j)$.

This is

$$u^{n+1}_j = u^n_j - C^n_{j-\frac{1}{2}}\,\delta u^n_{j-\frac{1}{2}} + D^n_{j+\frac{1}{2}}\,\delta u^n_{j+\frac{1}{2}}$$

with

$$C^n_{j-\frac{1}{2}} = \nu\left\{1 + \frac{1}{2}(1-\nu)\frac{\left[\phi_j\left(u^n_{j+1} - u^n_j\right) - \phi_{j-1}\left(u^n_j - u^n_{j-1}\right)\right]}{\delta u^n_{j-\frac{1}{2}}}\right\}$$

$$D^n_{j+\frac{1}{2}} = 0$$

The TVD criteria in this case require

$$0 \le C^n_{j-\frac{1}{2}} \le 1$$

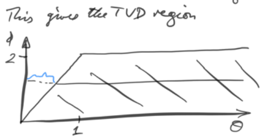
which for $0 \le \nu \le 1$ are satisfied if

$$\left|\frac{\phi(\theta_j)}{\theta_j} - \phi(\theta_{j-1})\right| \le 2$$

which reduces to $0 \le \left\{\dfrac{\phi(\theta)}{\theta}, \phi(\theta)\right\} \le 2$

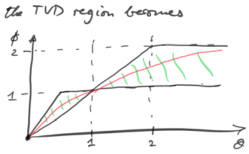
if we insist that $\phi(\theta) \ge 0$ and $\phi(\theta) = 0$ if $\theta \le 0$

This gives the TVD region



$\phi \equiv 0$ clearly gives 1st order upwind

whilst $\phi \equiv 1$ gives Lax-Wendroff but for 2nd order accuracy in smooth regions the TVD region becomes



and various choices of ϕ which remain in this region give 2nd order accurate TVD schemes

e.g. Van Leer Limiter $\phi(\theta) = \dfrac{\theta + |\theta|}{1 + |\theta|}$