Systems of nonlinear conservation laws

$$\underline{u}_t + \underline{f}(\underline{u})_x = 0 \quad , \underline{u}, \underline{f} \in \mathbb{R}^p$$

or $\underline{u}_t + A(\underline{u})\,\underline{u}_x = 0$,

$$A_{ij}(\underline{u}) = \frac{\partial f_i}{\partial u_j}(\underline{u}) \qquad i,j = 1,\dots,p$$

are tackled with local approximation of A, $\tilde{A}(\underline{u}_j^n)$ or $\tilde{A}(\underline{u}_{j-1}^n, \underline{u}_j^n)$ which must be consistent, thus $\tilde{A}(\underline{u},\underline{u}) = A(\underline{u})$

Then $\tilde{A} = X \Lambda X^{-1}$ and a set of decoupled linear advection equations results (similar to linear case).

e.g. shallow water equations

$$\underline{u} = \begin{bmatrix} v \\ \phi \end{bmatrix}, \quad \underline{f} = \begin{bmatrix} v^2/2 + \phi \\ v\phi \end{bmatrix}, \quad (\phi \geq 0)$$

so $A = \begin{bmatrix} v & 1 \\ \phi & v \end{bmatrix}$ has eigenvalues

$v + \sqrt{\phi} \leftrightarrow \begin{bmatrix} 1 \\ \sqrt{\phi} \end{bmatrix}$ ← 1st eigenvector

$v - \sqrt{\phi} \leftrightarrow \begin{bmatrix} 1 \\ -\sqrt{\phi} \end{bmatrix}$ ← 2nd eigenvector

Multidimensional conservation laws

$$u_t + f_x + g_y = 0$$

can be tackled with dimensional splitting method

i.e. from initial data

$$v_0(x,y) \simeq U^0 = \{U_{jk}^0\}$$

we can compute

$\hat{U}^1$ from solving the 1-dimensional problem $u_t + f_x = 0$ for 1 time step ①

then u^1 from solving $u_t + g_y = 0$ for 1 time step ② from initial data $\hat{U}^1$

and repeat. For linear advection

$$u_t + a u_x + b u_y = 0$$

which has exact solution

$$u(x,y,t) = u_0(x-at, y-bt)$$

the above splitting gives the correct answer since the solution of ① is

$$\hat{u}^1 = u_0(x-at, y)$$

Then the solution of ② is

$$\begin{aligned} u^1(x,y,t) &= u^1(x, y-bt, 0) \\ &= \hat{u}^1(x, y-bt, \epsilon) \\ &= u_0(x-at, y-bt). \end{aligned}$$

But for more general problems this successive solution of 1-dimensional problems gives an error which is proportional to t.

For finite difference approximation ① and ② are solved numerically, for a time step k ⇒ splitting error is $O(k)$ which is comparable to τ for a first order accurate scheme in time.

For a second order accurate scheme, a preferable splitting would have splitting error $O(k^2)$; one such is Strang splitting

$\hat{U}^1$ From solving $u_t + f_x = 0$ for time step $\frac{k}{2}$ from initial data U^0

$\tilde{U}^1$ " " $u_t + g_y = 0$ for time step k from initial data $\hat{U}^1$

U^1 " " $u_t + f_x = 0$ for time step $\frac{k}{2}$ from initial data $\tilde{U}^1$

and repeat.

Writing this as

$$U^1 = \underbrace{\mathcal{H}_{k/2}^x \mathcal{H}_k^y \underbrace{\mathcal{H}_{k/2}^x U^0}_{\hat{U}^1}}_{\tilde{U}^1}$$

we have

$$U^n = \mathcal{H}_{k/2}^x \left(\mathcal{H}_k^y \mathcal{H}_k^x\right)^{n-1} \mathcal{H}_k^y \mathcal{H}_{k/2}^x U^0$$

so unless solutions are required at intermediate times, the work (number of 1 step solution) required is comparable to the simple first order splitting which is

$$U^n = \left(\mathcal{H}_k^y \mathcal{H}_k^x\right)^n U^0$$

Otherwise: can often just interpret U_j^n as the approximation to

$$\frac{1}{h}\int_{x_j - \frac{h}{2}}^{x_j + \frac{h}{2}} u(x, t^n)\, dx$$

and so regard all we have done as a finite volume method. Higher dimensions can then just be considered as integration over boxes/cells as indicated earlier