Nonlinear Systems HT2021 — Sheet 2

1. Consider the equation
$$\ddot{x} = w - 2x + x^2$$
where $w \geq 0$ is a parameter.
   (i) Show that the evolution of x conserves a form of the energy and identify the potential function.
   (ii) From the potential function, sketch the phase portrait for $w = 0$. Identify important orbits.
   (iii) What happens as w increases? Find the critical value w such that the system does not support any periodic orbit.

2. Discuss the stability of the equilibria and limit cycles of
$$\begin{aligned}\dot{x} &= -y + x \sin r, \\ \dot{y} &= x + y \sin r\end{aligned}$$
where $r^2 = x^2 + y^2$.

3. The complex Landau equation
$$\dot{z} = az - b|z|^2 z,$$
arises in nonlinear stability theory. Here $z(t)$ is complex-valued and a, b are complex numbers (assume that $\mathrm{Re}(a) > 0$). Write the equation as a system of two real equations for $r(t)$ and $\theta(t)$ where $z = r(t)\mathrm{e}^{\mathrm{i}\theta(t)}$. Discuss the existence of periodic solutions in terms of the constants a and b.

4. A simple model for the motion of a glider is given by the equations
$$\begin{aligned}\dot{y} &= -\sin\theta - ay^2, \\ \dot{\theta} &= y - \frac{\cos\theta}{y},\end{aligned}$$
where y is the velocity, θ is the angle between the glider and the horizontal, and a is the ratio of the drag coefficient to lift coefficient. For $a = 0$ show that $V = y^3 - 3y\cos\theta$ is a conserved quantity and sketch the phase portrait. Interpret your result (What does the glider do? What is its path?).

   [*] For $a > 0$ (positive drag), linearise the system around its fixed points and discuss the stability. Again, interpret the results in terms its motion.

5. (i) Consider a vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$. Assume that $H = H(\mathbf{x})$ is a first integral ($\dot{H} = 0$). Let $\mathbf{x}_0$ be a fixed point. Prove that if $\mathbf{x}_0$ is a nondegenerate minimum of H, then $\mathbf{x}_0$ is stable.
   (ii) Let V be a $C^r (r \geq 1)$ function of $\mathbf{x} \in \mathbb{R}^n$. A *gradient vector field* or *gradient flow* is defined by
$$\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$$
   Show a gradient vector field cannot have periodic or homoclinic orbits (Hint: Use $V(x)$ as a Lyapunov function).

6. Show that the origin is a stable point of equilibrium for the nonlinear system

$$\dot{x} = y - x^3,$$
$$\dot{y} = -x^3,$$

but that it is an unstable point of equilibrium for the linearized system there [Hint: Consider Lyapunov functions of the form $V = x^m + cy^n$.]

7. By using ideas similar to Lyapunov's method, show that all trajectories of the **Lorenz system**

$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = \rho x - xz - y,$$
$$\dot{z} = xy - \beta z,$$

eventually enter and remain inside a large sphere S of the form $x^2 + y^2 + (z - \rho - \sigma)^2 = C$, for C sufficiently large.

8. Consider the system

$$\dot{x} = xy + ax^3 + xy^2,$$
$$\dot{y} = -y + bx^2 + x^2 y.$$

(i) Use an analysis of the dynamics on the centre manifold to show that the origin is asymptotically stable if $a + b < 0$ and unstable if $a + b > 0$.

(ii) What happens if $a + b = 0$? Is the origin stable or unstable?

9. A bead is free to slide without friction on a circular wire hoop of radius L. The hoop spins about its vertical axis with angular velocity ω. After nondimensionalisation, the equation governing the position $\theta(t)$ (measured from the bottom of the hoop) is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \sin\theta - \alpha \sin\theta \cos\theta = 0,$$

where $\alpha = \omega^2 L/g$.

(i) Discuss the behaviour of this system, as α increases from zero, from the point of view of bifurcation theory.

(ii) Write down an energy integral for the system. Find the smallest constant $v > 0$ (in terms of the parameters) such that, if initially $\theta = \pi/2$, $|\dot{\theta}| > v$, then the bead will continually encircle the hoop in one direction.

(iii) [*] What happens if linear damping is added to the system (that is, $-\mu\dot{\theta}$ is added on the equation's RHS with $\mu > 0$)? (NB: There is a real zoo of possible bifurcations in this system. A simple and good starting point is to find the critical rotation speed at which $\theta = 0$ becomes unstable. Describe this bifurcation).