Nonlinear Systems HT2021 — Sheet 3

1. Consider the system

$$\dot{x} = y - x - x^2,$$
$$\dot{y} = \mu x - y - y^2.$$

Find the value of μ for which there is a bifurcation at the origin. Find the evolution equation on the extended centre manifold correct to quadratic terms in the Taylor expansion and determine the type of bifurcation.

2. Consider the system

$$\begin{aligned} \dot{x} &= \mu x + y + \sin x, \\ \dot{y} &= x - y. \end{aligned} \tag{1}$$

Show that a bifurcation occurs at the origin of this system. Use the extended centre manifold to classify it. Draw the local (close to the origin) phase portraits before and after the bifurcation.

3. Consider the 1D map

$$x_{n+1} = f(x_n),$$

and assume that it supports a p-periodic orbit $\{x_1, x_2, \ldots, x_p\}$ such that $x_i \neq x_j \ \forall i, j \in \{1, \ldots, p\}$ with $i \neq j$, and $x_{p+1} = x_1$. Show that the stability of this orbit is determined by the multiplier

$$\lambda = \prod_{i=1}^{p} f'(x_i),$$

whenever $|\lambda| \neq 1$.

4. Consider the map T defined by

$$x_{n+1} = y_n,$$
$$y_{n+1} = -x_n + 7y_n - y_n^3.$$

(i) Find the fixed points, determine their stability and compute their local stable and unstable subspaces.

(ii) Show that T admits 3 orbits of period 2, all within the square $S = \{(x, y) \mid |x| \leq 3, |y| \leq 3\}$.

(iii) Show that every orbit starting outside S tends to infinity for either $n \to \infty$ or $n \to -\infty$ and hence there are no periodic orbits outside S.
[Hint: Divide the complement of S into the regions

$$R_1 = \{(x, y) \mid x + y \geq 0, y > 3\}, \qquad R_2 = \{(x, y) \mid x + y < 0, y \leq -3\},$$

$$R_3 = \{(x, y) \mid x + y \geq 0, x > 3\}, \qquad R_4 = \{(x, y) \mid x + y < 0, x \leq -3\}.$$

Show a point in R_1 is mapped to a point in R_2 and vice versa, and that for such points $|y_{n+1}| > |y_n|$. For R_3 and R_4 consider the inverse map.]

(The map T admits infinitely many periodic orbits but the proof is somewhat more involved.)

5. Consider the system

$$\ddot{x} + x - \epsilon x^2 = 0$$

for an asymmetric spring. Find the equilibrium points, and for which values of ϵ each exists. Find for which values of ϵ each is stable, and classify the type of point. Sketch the bifurcation diagram in the (ϵ, x)-plane.

Consider the periodic orbit satisfying $\dot{u}(0) = 0$, $u(0) = a$. Use the Poincaré-Lindstedt method to find the expansion of the frequency of this orbit up to [and including] terms of $O(\epsilon^2)$.

6. In general relativity the equation of the orbit of a planet with polar coordinates (r, θ) is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{1}{\ell} + \epsilon \ell u^2,$$

where $u = 1/r$, the sun is fixed at the origin $r = 0$, $\ell = h^2/GM$, and $\epsilon = 3GM/c^2\ell$. Here G is the gravitational constant, c is the velocity of light, M is the mass of the sun, and h is the angular momentum per unit mass of the planet about the sun.

Show that there is a centre at $(u_N, 0)$ and a saddle point at $(u_\tau, 0)$ in the $(u, \mathrm{d}u/\mathrm{d}\theta)$-plane, where

$$u_N(\epsilon) = \frac{1}{\ell} + \frac{\epsilon}{\ell} + O(\epsilon^2), \qquad \text{and} \qquad u_\tau(\epsilon) = \frac{1}{\epsilon\ell} - \frac{1}{\ell} + O(\epsilon),$$

as $\epsilon \to 0$. Sketch the phase portrait and identify the region of orbits representing solutions which are periodic functions of θ.

Define $\phi = \omega\theta$, where $2\pi/\omega$ is the period of an orbit, and assume that

$$\omega(\epsilon) = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots .$$

Hence show that $\omega_0 = 1$ and $\omega_1 = -1$. Deduce that the planet is at *perihelium* (i.e. that its distance from the sun is a local minimum) at successive angles θ differing by $2\pi/\omega = 2\pi + 2\pi\epsilon + O(\epsilon^2)$ as $\epsilon \to 0$.

[This gives the precession of the perihelium of the planet by the angle $2\pi\epsilon = 6\pi GM/c^2\ell$ approximately each revolution, where ℓ is close to the mean radius of the orbit. This result is used in one of the classic tests of Einstein's general theory of relativity.]