Nonlinear Systems HT2021 — Sheet 4

1. The Brusselator is a simple model for a hypothetical chemical oscillator, named after the home of the scientists who proposed it. In dimensionless form, the dynamics is given by

$$\begin{aligned} \dot{x} &= a - (b+1)x + x^2 y\,, \\ \dot{y} &= bx - x^2 y\,, \end{aligned} \tag{1}$$

where $a, b > 0$ are parameters and $x, y \geq 0$ are dimensionless concentrations. Consider the case $a = 1$. Show that a Hopf bifurcation occurs at some parameter value $b = b_c$, where you should determine b_c. Find an approximation of the amplitude and the period of the limit cycle close to $b = b_c$.

2. Analyze the Hopf bifurcation for the system

$$\ddot{x} + \mu\dot{x} + x + x^2\dot{x} + x^3 = 0.$$

with μ as the bifurcation parameter.

3. Consider the map

$$x_{n+1} = (1+\mu)x_n - \mu x_n^2,$$

for $\mu \geq 0$.

   (i) Find the fixed points and analyse their stabilities.

   (ii) Find the period-2 cycles and analyse their stabilities.

   (iii) Draw the bifurcation diagram in the (μ, x)-plane.

   (iv) [$*$] Verify your results by computing (numerically) the full bifurcation diagram of x_n for $0 < \mu \leq 3$.

4. Consider the mapping $F : [0,1] \to [0,1]$

$$F(x) = 2x \bmod 1$$

   (i) Show that if $x \in [0,1]$ has the binary expansion

$$x = 0.s_1 s_2 \ldots = \sum_{i=1}^{\infty} \frac{s_i}{2^i}$$

   with $s_i \in \{0,1\}$, then

$$F^n(x) = 0.s_{n+1}s_{n+2}\ldots.$$

   (ii) Prove the existence of a countable infinity of periodic orbits.

   (iii) Prove the existence of a uncountable infinity of nonperiodic orbits.

   (iv) Show that there is a dense orbit.

   (v) Show that the system has sensitive dependence to initial conditions.

5. Consider the perturbed Duffing equation

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + \epsilon(\alpha y + \beta x^2 y). \end{aligned}$$

Draw the phase portrait for $\epsilon = 0$. Use Melnikov's analysis to show that there can be no transverse homoclinic point. Find a condition on the parameters α and β such that two homoclinic orbits exist for ϵ small enough. Draw the perturbed phase portrait for $\epsilon\alpha < 0$.

[You may find the following identities useful:

$$\int_{-\infty}^{\infty} \operatorname{sech}^2\tau \tanh^2\tau \, d\tau = \frac{2}{3}, \qquad \int_{-\infty}^{\infty} \operatorname{sech}^4\tau \tanh^2\tau \, d\tau = \frac{4}{15}. \qquad]$$

6. If the shape of a bubble is given by $r = r(\theta, t)$ then a scalar measure of the deviation of the bubble from sphericity is

$$x = \int_0^\pi r(\theta, t) P_2(\cos\theta) \sin\theta \, \mathrm{d}\theta$$

where P_2 is a Legendre polynomial.

The evolution of x, in the presence of a time-periodic, axisymmetric, uniaxial extensional flow of a fluid, in dimensionless variables, is given by

$$\ddot{x} = w - 2x + x^2 - \epsilon(\mu\dot{x} + \delta\cos\omega t),$$

where

$$w = \frac{4 \times 144 \times 755}{67^2 W_0 a^2}, \qquad W_0 = \frac{2\rho E_0^2 a^3}{\gamma}, \qquad V = \frac{4}{3}\pi a^3,$$

where V is the bubble volume, γ is the surface tension, E_0 the principal strain rate, ρ the fluid density and μ its viscosity (W_0 is the Weber number). After writing $x = 1 - \sqrt{1 - w} - u$, the system reads:

$$\ddot{u} + \omega_0^2 u + u^2 = \epsilon(-\mu\dot{u} + \delta\cos\omega t),$$

where $\omega_0^2 = 2\sqrt{1 - w}$.

For this system, compute the Melnikov function for the existence of transverse homoclinic points. Determine the critical forcing amplitude δ for a Smale horseshoe to exist. Hence show that the optimal forcing frequency for bubble breakup is $\omega = k\omega_0/\pi$ where

$$k = 2\tanh k.$$

[You may find the following identities useful:

$$\int \frac{\mathrm{d}u}{(u+a)\sqrt{a - 2u}} = -\frac{2}{\sqrt{3a}} \tanh^{-1}\left(\sqrt{\frac{a - 2u}{3a}}\right).$$

$$\int_{-\infty}^{\infty} \operatorname{sech}^4\tau \tanh^2\tau \, \mathrm{d}\tau = \frac{4}{15}.$$

$$\int_{-\infty}^{\infty} \operatorname{sech}^2\tau \tanh\tau \, \sin(a\tau) \, \mathrm{d}\tau = \frac{a^2\pi}{2} \operatorname{csch}\left(\frac{a\pi}{2}\right). \qquad]$$