

Waves and Compressible Flow

Lecture 4

Example: Stokes waves on a free surface

P.1

- Consider small amplitude waves on surface of constant density fluid, e.g. ripples on a pond.



- $\rho = \text{constant} \Rightarrow \nabla \cdot \underline{u} = 0$
Assume irrotational $\Rightarrow \exists \phi$ s.t. $\underline{u} = \nabla \phi$ } $\Rightarrow \nabla^2 \phi = 0$ ①

- Euler's equation $\Rightarrow$ Bernoulli's equation $\frac{p}{\rho} + \phi_t + \frac{1}{2}|\underline{u}|^2 + gz = F(t)$ ②

- Note ①-② hold in $-h < z < \eta(x, y, t)$, $-\infty < x, y < \infty$.

- Zero normal velocity on substrate $\Rightarrow \underline{u} \cdot \underline{k} = 0$ or $\phi_z = 0$ on $z = -h$ ③
- On free surface $z = \eta$ need two BCs because η is unknown.
- Kinematic BC: $\frac{D}{Dt}(z - \eta) = 0$ on $z = \eta \Rightarrow \phi_z = \eta_t + \phi_x \eta_x + \phi_y \eta_y$ on $z = \eta$ ④
- Dynamic BC: $p = p_{\text{atm}}$ on $z = \eta \Rightarrow \phi_t + \frac{1}{2} |\nabla \phi|^2 + g\eta = 0$ on $z = \eta$ ⑤
 (② with $F = \frac{p_{\text{atm}}}{\rho} \omega \log$)
- ①-⑤ nonlinear so hard to solve. To make progress we assume disturbance is small so we can linearize the problem.

- Neglecting quadratic and h.o.t. in (4) suggests $\phi_z = \eta_t$ on $z = \eta$.
- But a Taylor expansion gives $\phi_z|_{z=\eta} = \phi_z|_{z=0} + \eta \phi_{zz}|_{z=0} + \text{h.o.t.}$,
so linearizing (4) in fact gives $\phi_z = \eta_t$ on $z = 0$.
- Similarly, (5) gives $\phi_t + g\eta = 0$ on $z = 0$ upon linearization.
- Hence, linearized problem is $\nabla^2 \phi = 0$ in $-h < z < 0$ (1L)
with $\phi_z = 0$ on $z = -h$ (2L)
and $\phi_z = \eta_t, \phi_t + g\eta = 0$ on $z = 0$ (3L)

- Seek sinusoidal travelling wave solution $\eta = A e^{i(kx - \omega t)}$, where real part is understood, A is constant complex amplitude, ω is frequency and k is wavenumber; note wavelength $\lambda = \frac{2\pi}{k}$ and phase speed $c_p = \frac{\omega}{k}$.

- Consistent form for ϕ is $\phi = f(z) e^{i(kx - \omega t)} \Rightarrow$ $\textcircled{1L} \quad f'' - k^2 f = 0$ for $-h < z < 0$.

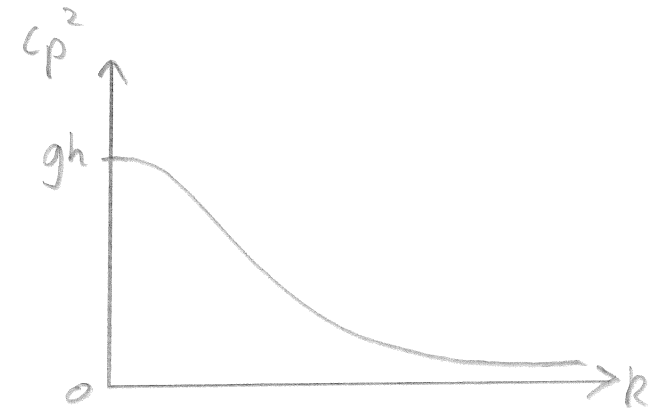
- Since $\textcircled{2L} \Rightarrow f'(-h) = 0$, we deduce that $f = B \cosh k(z+h)$ ($B \in \mathbb{C}$)

- $\textcircled{3L} \Rightarrow f'(0) = -i\omega A, -i\omega f(0) + gA = 0 \Rightarrow$

$$\underbrace{\begin{bmatrix} i\omega & k \sinh(kh) \\ g & -i\omega \cosh(kh) \end{bmatrix}}_M \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- M invertible $\Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \phi = 0, \eta = 0$, i.e. the trivial solution.
- Hence, a nontrivial solution can only exist if $\det(M) = 0 \Leftrightarrow \omega^2 = gk \tanh(kh)$, which is a dispersion relation (as it relates ω and k).

- Phase speed c_p s.t. $c_p^2 = \frac{\omega^2}{k^2} = \frac{g}{k} \tanh(kh)$,
so long waves (small k) travel faster than short waves (large k).



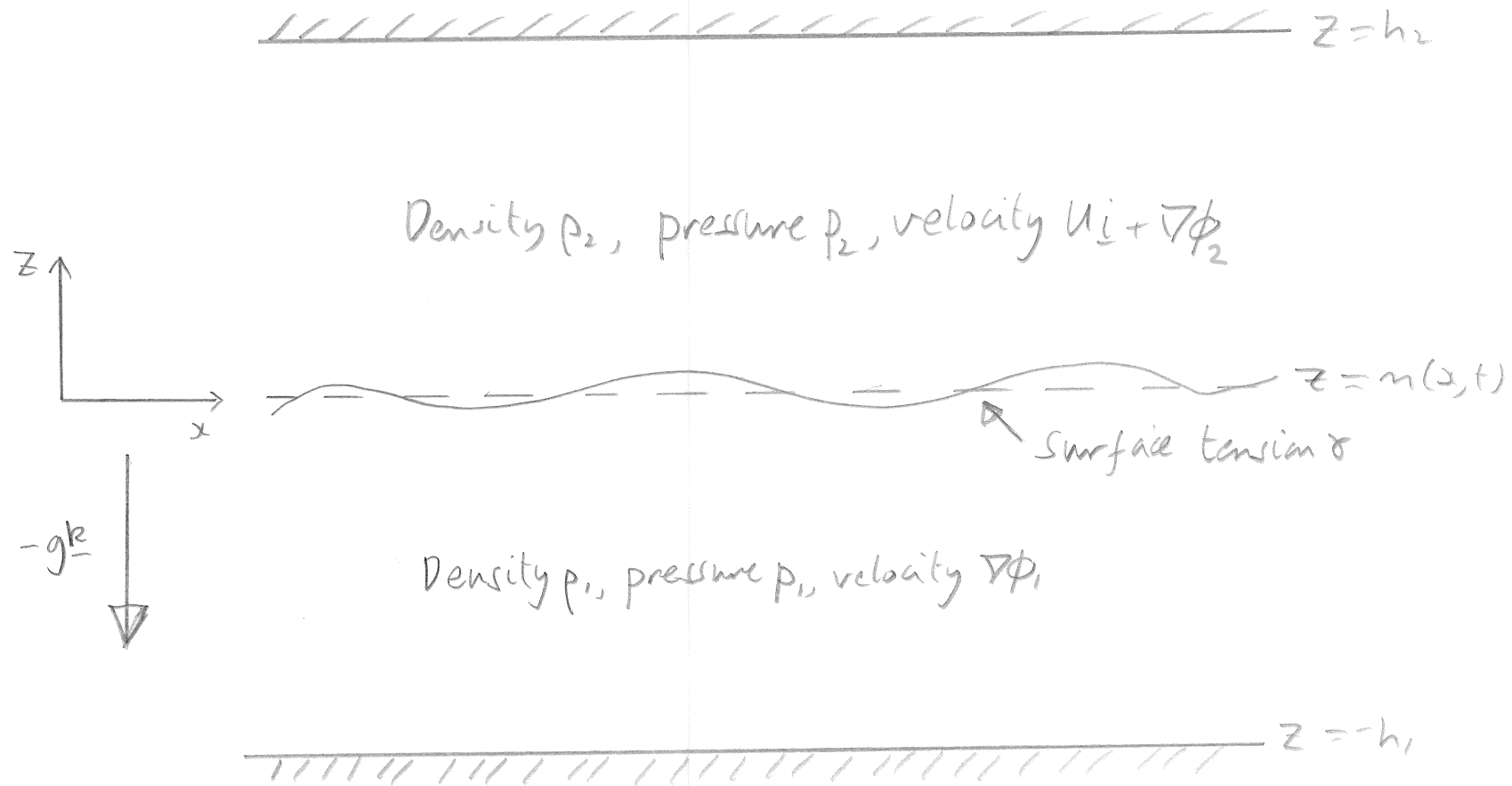
- Waves with different wavelengths travel at different speeds — they are called dispersive, cf. acoustic waves which have constant phase speed $c_p = c_0$.

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Example: common generalizations

p.6



Governing equations

p.7

- $\nabla^2 \phi_1 = 0$, $\frac{p_1}{\rho_1} + \phi_{1t} + \frac{1}{2}|\nabla \phi_1|^2 + gz = F_1(t)$ for $-h_1 < z < \eta$;
 $\nabla^2 \phi_2 = 0$, $\frac{p_2}{\rho_2} + \phi_{2t} + \frac{1}{2}|\underline{U} + \nabla \phi_2|^2 + gz = F_2(t)$ for $\eta < z < h_2$.
- $\phi_{1z} = 0$ on $z = -h_1$; $\phi_{2z} = 0$ on $z = h_2$.
- $\phi_{1z} = \eta_t + \phi_{1z}\eta_x$, $\phi_{2z} = \eta_t + (U + \phi_{2z})\eta_x$, $p_2 - p_1 = \gamma K$ all on $z = \eta$,
where the curvature $K = \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}$.
- See part A fluids and online notes for a derivation of the dynamic BC.

Linearized governing equations

1 p. 8

- Base state is $\phi_1 = 0, \phi_2 = 0, \eta = 0$, so convenient to set $F_1 = 0, F_2 = \frac{1}{2}U^2$.
- $\nabla^2 \phi_1 = 0$ in $-h_1 < z < 0$; $\nabla^2 \phi_2 = 0$ in $0 < z < h_2$.
- $\phi_{1z} = 0$ on $z = -h_1$; $\phi_{2z} = 0$ on $z = h_2$.
- $\phi_{1z} = \eta_t, \phi_{2z} = \eta_t + U\eta_x, -\rho_2(\phi_{2t} + U\phi_{2x} + g\eta) + \rho_1(\phi_{1t} + g\eta) = \gamma\eta_{xx}$ all on $z = 0$; be careful to substitute for p_1 and p_2 before linearizing.

- Seek a sinusoidal travelling wave solution of the form

$$\eta = A e^{i(kx - \omega t)}$$

$$\phi_1 = B e^{i(kx - \omega t)} \cosh k(z + h_1)$$

$$\phi_2 = C e^{i(kx - \omega t)} \cosh k(z - h_2)$$

where real part is understood, the complex amplitudes $A, B, C \in \mathbb{C}$

and ϕ_1, ϕ_2 have been chosen to satisfy Laplace's equation and

the boundary conditions on $z = -h_1$ and $z = h_2$.

• BCs on $z = 0 \Rightarrow M \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for some 3×3 matrix M .

• Nontrivial solutions $\Leftrightarrow \det(M) = 0 \Leftrightarrow \omega$ and k are related by

$$\omega^2 p_1 \coth(kh_1) + (\omega - Uk)^2 p_2 \coth(kh_2) = (p_1 - p_2)gk + \gamma k^3$$

• Quadratic equation for $\omega(k)$ of the form $a(k)\omega^2 + 2b(k)\omega + c(k) = 0$,

where $a(k) = p_1 \coth(kh_1) + p_2 \coth(kh_2)$, $2b(k) = -2Up_2 k \coth(kh_2)$ and

$$c(k) = U^2 k^2 p_2 \coth(kh_2) - (p_1 - p_2)gk - \gamma k^3.$$

• Roots $\omega_{\pm}(k) = \frac{-b \pm \sqrt{b^2 - ac}}{a}$ tough to analyse, so consider two important limiting cases.

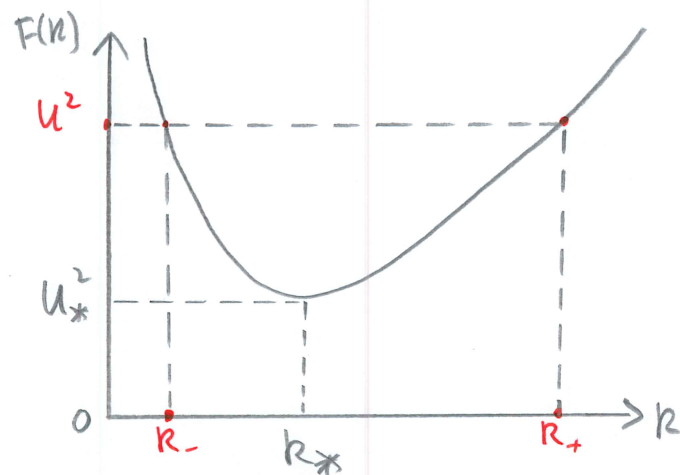
Example: $h_1 = \infty, h_2 = \infty, U = 0$

- In this case, $\omega^2 = \frac{(\rho_1 - \rho_2)g + \gamma k^2}{\rho_1 + \rho_2} |k|$
- $\rho_1 > \rho_2$ $\Rightarrow$ RHS $> 0 \Rightarrow$ two real roots $\Rightarrow$ disturbance stable
- $\rho_1 < \rho_2$ $\Rightarrow$ RHS < 0 for $|k| < k_c = \left(\frac{(\rho_2 - \rho_1)g}{\gamma} \right)^{1/2}$, and for such wavenumbers, $\omega = \pm i\omega_I(k)$ with $\omega_I \in \mathbb{R}$, giving $\eta = A e^{ikx \pm \omega_I t} \Rightarrow$ disturbance unstable when denser fluid above lighter fluid.
- This is known as the Rayleigh-Taylor instability.

Example: $h_1 = \infty, h_2 = \infty, U \neq 0, \rho_1 > \rho_2$

• New discriminant $b^2 - ac = 4\rho_1\rho_2 k^2 \left[\underbrace{\frac{\rho_1 + \rho_2}{\rho_1\rho_2} \frac{(\rho_1 - \rho_2)g + \gamma k^2}{|k|}}_{= F(k)} - U^2 \right]$

• Sketch $F(k)$:



Minimum at

$$k_* = ((\rho_1 - \rho_2)g/\gamma)^{1/2}$$

$$U_*^2 = \frac{2(\rho_1 + \rho_2)}{\rho_1\rho_2} (\gamma(\rho_1 - \rho_2)g)^{1/2}$$

• $U^2 < U_*^2 \Rightarrow U^2 < F(k) \forall k \Rightarrow b^2 - ac > 0 \forall k \Rightarrow$ stable.

• $U^2 > U_*^2 \Rightarrow U^2 > F(k)$ for a band $k \in (k_-, k_+) \Rightarrow b^2 - ac < 0$ and unstable for such k .

This is the Kelvin-Helmholtz instability — onset of “sea horses”