Question Sheet 1

1. Consider *barotropic flow*, in which the pressure is a known function of the density, $p = P(\rho)$. Assume also that the body force is *conservative*, that is $\boldsymbol{g} = -\boldsymbol{\nabla}\chi$ for some potential function χ. Throughout this question, take care that you do *not* assume that ρ is constant.

   (a) Show that the momentum equation may be written in the form

   $$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{u} = \boldsymbol{\nabla}\psi$$

   for some scalar function ψ (which you should define). Hence show that the *vorticity* $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ satisfies

   $$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}}{\rho}.$$

   Deduce that, in two-dimensional barotropic fluid, $\boldsymbol{\omega}/\rho$ is conserved following the flow.

   (b) Let $C(t)$ be a simple closed curve that is convected with the fluid. Define the *circulation* Γ around C by

   $$\Gamma = \oint_C \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x}.$$

   By transforming to Lagrangian variables, or otherwise (*e.g.* Acheson exercise 5.2), show that Γ is independent of t; this is known as *Kelvin's Circulation Theorem.* Deduce that a flow which is initially *irrotational* (*i.e.* $\boldsymbol{\omega} = \boldsymbol{0}$ at $t = 0$) remains irrotational for all time.

   (c) Assuming the flow is irrotational, define a *velocity potential* ϕ such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. Deduce *Bernoulli's equation* for unsteady irrotational barotropic flow, namely

   $$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \int \frac{P'(\rho)}{\rho}\,\mathrm{d}\rho + \chi = F(t).$$

   Explain carefully how an appropriate definition of ϕ allows $F(t)$ to be chosen arbitrarily.

2. Show that incompressible flow with constant density ρ relative to axes rotating with constant angular velocity $\boldsymbol{\Omega}$ is governed by the equations

   $$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \qquad \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}P,$$

   where the *reduced pressure* P is to be defined, and body forces (*e.g.* gravity) have been neglected. Deduce that small steady perturbations to the steady state $\boldsymbol{u} = \boldsymbol{0}$, $P = \text{const}$ satisfy

   $$\boldsymbol{\nabla}\cdot\boldsymbol{u}' = 0, \quad 2\boldsymbol{\Omega}\times\boldsymbol{u}' = -\frac{1}{\rho}\boldsymbol{\nabla}P'.$$

   Hence explain the following observations (from the Met. Office website):

   - the closer the isobars, the stronger the wind;
   - the wind blows almost parallel to the isobars;
   - the direction of the wind is such that, if you stand with your back to the wind in the northern hemisphere, the pressure is lower on your left than on your right.

[*Hint: model the earth as the plane $z = 0$ rotating with angular velocity $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{e}}_z$.*]

3. A non-conducting inviscid gas, initially at rest with pressure p_0 and density ρ_0, is heated internally at a small rate $q(\boldsymbol{x}, t)$ per unit mass. You may assume that body forces are negligible, and that the gas obeys the ideal gas law. By linearising the equations for conservation of mass, momentum and energy, show that the pressure, density and velocity perturbations (denoted with primes) approximately satisfy

$$\frac{\partial \rho'}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}' = 0, \qquad \rho_0 \frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{\nabla} p' = \boldsymbol{0}, \qquad \frac{\partial p'}{\partial t} - c_0^2 \frac{\partial \rho'}{\partial t} = \rho_0(\gamma - 1)q,$$

where γ is the ratio of specific heats and $c_0^2 = \gamma p_0/\rho_0$. Deduce that ρ' satisfies

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \nabla^2 \rho'\right) = (\gamma - 1)\rho_0 \nabla^2 q.$$

If $q = \mathrm{Re}\left\{A\mathrm{e}^{\mathrm{i}(kx-\omega t)}\right\}$, show that the oscillations forced in ρ' are out of phase with q and have amplitude

$$\frac{(\gamma - 1)\rho_0 |A| k^2}{\omega|\omega^2 - c_0^2 k^2|}.$$

What happens when $\omega \to c_0 k$?

4. Incompressible fluid of uniform density ρ_1 lies at rest in $z < 0$, beneath incompressible fluid of uniform density ρ_2 in $z > 0$ moving with uniform speed U in the x-direction. The interface between the fluids is subject to surface tension γ.

Small-amplitude disturbances perturb the interface to $z = \eta(x, t)$ and propagate as waves. If variables in the lower and upper fluids are denoted by suffices 1 and 2 respectively, derive the linearised boundary conditions

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \eta}{\partial t}, \quad \frac{\partial \phi_2}{\partial z} = \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x}, \quad \rho_1\left(\frac{\partial \phi_1}{\partial t} + g\eta\right) - \rho_2\left(\frac{\partial \phi_2}{\partial t} + U\frac{\partial \phi_2}{\partial x} + g\eta\right) = \gamma\frac{\partial^2 \eta}{\partial x^2},$$

on $z = 0$. Show that waves with $\eta = \mathrm{Re}\left\{A\mathrm{e}^{\mathrm{i}(kx-\omega t)}\right\}$, where $k > 0$, are possible provided

$$\rho_1(gk - \omega^2) + \gamma k^3 = \rho_2\left[(\omega - Uk)^2 + gk\right].$$

Deduce that such a disturbance is unstable if

$$U^2 > \left(\frac{\rho_1 + \rho_2}{\rho_1 \rho_2}\right)\left(\frac{(\rho_1 - \rho_2)g + \gamma k^2}{k}\right).$$

If $\rho_1 > \rho_2$, what is the minimum speed U required for the interface to become unstable? What is the wavelength $\lambda = 2\pi/k$ of the waves that first become unstable? [*Hint: think about how the critical speed just found depends on wave number k*]

If $\rho_2 > \rho_1$, show that the interface is unstable even at zero speed, but only for wavenumbers smaller than a critical value k_*, which you should find.