

## B5.4 Waves & Compressible Flow

### Question Sheet 3 Solutions to Optional Questions

(i) (a) Define

$$I = \int_{-R}^R e^{\pm is^2} ds, \quad (1)$$

and deform the contour into three components (see the figure):

$$\gamma_1 : s = Re^{i(\theta+\pi)}, \quad \gamma_2 : s = \frac{(1 \pm i)t}{\sqrt{2}}, \quad \gamma_3 : s = Re^{i\theta}. \quad (2)$$

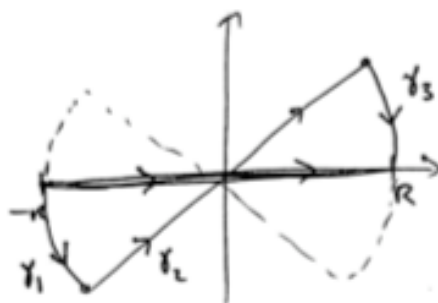


Figure 1: Contours for question (i)

Then,

$$\begin{aligned} I &= \int_0^{\pm\pi/4} e^{iR^2(\cos 2\theta + i \sin 2\theta)} 2iRe^{i(\theta+\pi)} d\theta + \int_{-R}^R e^{(\pm i)^2 t^2} \frac{(1 \pm i)}{\sqrt{2}} dt \\ &\quad + \int_0^{\pm\pi/4} e^{\pm iR^2(\cos 2\theta + i \sin 2\theta)} 2iRe^{i\theta} d\theta, \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (3)$$

Now,

$$I_1 = \int_{-\infty}^{\infty} e^{-t^2} dt \frac{(1 \pm i)}{\sqrt{2}} = \sqrt{\pi} \frac{(1 \pm i)}{\sqrt{2}} = (1 \pm i) \sqrt{\frac{\pi}{2}}. \quad (4)$$

Also,

$$I_2 = -2 \int_R^{\infty} e^{-t^2} dr \frac{(1 \pm i)}{\sqrt{2}}, \quad (5)$$

so that (substituting  $t = R + s$ ),

$$|I_2| \leq 2e^{-R^2} \int_0^{\infty} e^{-Rs} e^{-s^2} ds \leq 2e^{-R^2} \frac{\sqrt{\pi}}{2} = \mathcal{O}(e^{-R^2}). \quad (6)$$

Finally,

$$I_3 = 4iR \int_0^{\pm\pi/4} e^{\pm iR^2(\cos \theta + i \sin 2\theta)} e^{i\theta} d\theta, \quad (7)$$

so that (using  $\sin 2\theta \geq 2\theta/\pi$ ),

$$\begin{aligned}
|I_3| &\leq 4R \int_0^{\pm\pi/4} \left| e^{\pm R^2 \sin 2\theta} \right| d\theta \\
&\leq 4R \int_0^{\pi/4} e^{-R^2 \cdot 2\theta/\pi} d\theta, \\
&\leq 4R \frac{\pi}{2R^2} \left[ e^{-2R^2\theta/\pi} \right]_0^{\pi/4}, \\
&= \mathcal{O}(1/R).
\end{aligned} \tag{8}$$

Hence,

$$I = (1 \pm i) \sqrt{\frac{\pi}{2}} + \mathcal{O}(1/R). \tag{9}$$

(b) Define

$$I(t) = \int_a^b f(k) e^{ikt} dk. \tag{10}$$

We integrate by parts by setting  $u = e^{ikt}/it$  and  $v = f(k)$  to give

$$I(t) = \left[ \frac{1}{it} e^{ikt} f(k) \right]_a^b - \int_a^b \frac{1}{it} e^{ikt} f'(k) dk. \tag{11}$$

Therefore

$$|I(t)| \leq \frac{1}{t} (|f(a)| + |f(b)|) + \frac{1}{t} (b-a) \sup_k \{|f'(k)|\}, \tag{12}$$

and providing  $f$  and  $f'$  are bounded,

$$|I(t)| \leq \frac{M}{t} \quad \text{for some } M. \tag{13}$$

Therefore  $I(t) = \mathcal{O}(1/t)$  as  $t \rightarrow \infty$ .

(c) See lecture notes.

(ii) The Fourier transform of  $\epsilon/(x^2 + \epsilon^2)$  is given by

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{\epsilon}{x^2 + \epsilon^2} e^{-ikx} dx. \tag{1}$$

The integrand has poles at  $x = \pm i\epsilon$ , and can be written in the form

$$\frac{\epsilon e^{-ikx}}{(x + i\epsilon)(x - i\epsilon)}. \tag{2}$$

Therefore, the residues are

$$\frac{\epsilon e^{\pm k\epsilon}}{\pm 2i\epsilon} = \mp \frac{1}{2} i e^{\pm k\epsilon}. \tag{3}$$

For  $k > 0$  we close the contour in the lower half plane, and for  $k < 0$  we close the contour in the upper half plane (see figure 2). Jordan's Lemma tells us that the semi-circular contour doesn't contribute as  $R \rightarrow \infty$ , so we just pick up the residue at the pole. Therefore,

$$\hat{f}(k) = \begin{cases} (-2\pi i) \left( \frac{i}{2} e^{-k\epsilon} \right) & \text{for } k > 0, \\ (2\pi i) \left( -\frac{i}{2} e^{k\epsilon} \right) & \text{for } k < 0, \end{cases} \tag{4}$$

so that

$$\hat{f}(k) = \pi e^{-|k|\epsilon}. \quad (5)$$

Also note that the case  $k = 0$  follows by continuity, or by direct integration:

$$\hat{f}(0) = \int_{-\infty}^{\infty} \frac{\epsilon}{x^2 + \epsilon^2} dx = \left[ \arctan\left(\frac{x}{\epsilon}\right) \right]_{-\infty}^{\infty} = \pi. \quad (6)$$

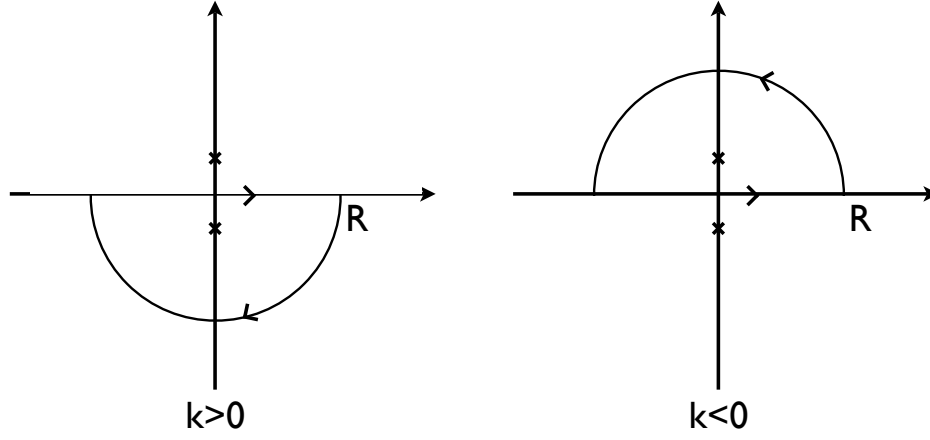


Figure 2: The contours for Question (ii).

Now we assume that fluid occupying the half-space  $z < 0$  starts from rest with the initial free surface profile  $\eta_0(x) = -a\epsilon/\pi(x^2 + \epsilon^2)$ . The solution in the lecture notes gives

$$\hat{\eta}(k, t) = \hat{\eta}_0(k) \cos(w(k)t), \quad (7)$$

with  $w(k) = \sqrt{g|k|}$ . Using the result above we have that  $\hat{\eta}_0(k) = -ae^{-\epsilon|k|}$ . We invert the transform to get

$$\begin{aligned} \eta(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\eta}(k, t) e^{ikx} dk, \\ &= -\frac{a}{2\pi} \int_{-\infty}^{\infty} e^{-\epsilon|k|} \cos\left(t\sqrt{g|k|}\right) e^{ikx} dk, \\ &= I_+ + I_-, \end{aligned} \quad (8)$$

where

$$I_{\pm} = -\frac{a}{4\pi} \int_{-\infty}^{\infty} e^{-\epsilon|k|} e^{i\left(kx/t \pm \sqrt{g|k|}\right)} dk. \quad (9)$$

The method of stationary phase tells us that the main contribution to the integral comes from  $\psi'(k) = 0$  where

$$\psi(k) = \frac{kx}{t} \pm \sqrt{g|k|}, \quad (10)$$

*i.e.* where

$$\frac{x}{t} \pm \frac{\sqrt{g|k^*|}}{2k^*} = 0 \quad \Rightarrow \quad k^* = \mp \frac{gt^2}{4x^2}. \quad (11)$$

Further,

$$\psi''(k^*) = \mp \frac{\sqrt{g|k^*|}}{4k^{*2}} = \mp \frac{\sqrt{g}}{4} \frac{8x^3}{g^{3/2}t^3} = \mp \frac{2x^3}{gt^3}. \quad (12)$$

Therefore the method of stationary phase gives

$$I_{\pm} \sim -\frac{a}{4\pi} e^{-\epsilon|k^*|} e^{i(\psi(k^*)t + \frac{\pi}{4} \text{sgn}(\psi''(k^*)))} \sqrt{\frac{2\pi}{|\psi''(k^*)|t}}. \quad (13)$$

Now, for  $\epsilon$  sufficiently small,  $e^{-\epsilon|k^*|} \sim 1$ , so

$$\begin{aligned} I_{\pm} &\sim -\frac{a}{4\pi} e^{i\left[\mp \frac{gt^2}{4x} \pm \frac{gt^2}{2x} \mp \frac{\pi}{4}\right]} \sqrt{\frac{2\pi gt^3}{2x^3t}} \\ &\sim -\frac{at}{4} e^{\pm i\left[\frac{gt^2}{4x} - \frac{\pi}{4}\right]} \sqrt{\frac{g}{\pi x^3}}. \end{aligned} \quad (14)$$

Therefore,

$$\eta(x, t) = I_+ + I_- \sim -\frac{at}{2} \sqrt{\frac{g}{\pi x^3}} \cos\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right). \quad (15)$$