- A quick question about supersonic flow past a thin wing. On page 60 of the notes, where we're finding the value of ϕ in each of the 6 regions, in regions 3 and 6, is ϕ supposed to be 0 or a constant (so that we have continuity between regions 2 and 3, 5 and 6)? The notes say that ϕ should be 0 (so that they are zones of silence) but in the lectures, you imposed continuity so that ϕ are non-zero constants instead, so I was just wondering which one I should use? Can we still say that there are zones of silence if ϕ is non-zero?

The perturbed velocity \nabla \phi is the same in both cases and defined except on the Mach lines. Hence, the zones of silence where \nabla \phi vanishes are identical in each case. As mentioned in lecture 16, the Mach lines are really shocks (and weak in the sense that the jump in entropy is small because the jump in velocity is small), so the two solutions are really different weak solutions. I prefer the one in lectures as it treats the shocks the same, i.e. \phi is continuous across them, with the discontinuity in \phi being across the wake at y=0, x > a. (Note that there must be a discontinuity in \phi around the wing because the circulation is non-zero.) A better description of the physics is given in section 4.6.3 of Ockendon & Ockendon on the reading list. My aim in lectures was more modest: I tried to turn it into a problem as close as possible to the ones you saw on semi-linear hyperbolic PDEs in part A DEs 1.

2012

1. (a) Waves in a linearly elastic solid satisfy Navier's equation

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

where the stress tensor is given by

$$\sigma_{ij} = \lambda(\nabla \cdot \mathbf{u})\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

for material constants λ, μ and we use the summation convention that a repeated index indicates summation over that index.

(i) Show that Navier's equation may also be written in vector form as

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \wedge (\nabla \wedge \mathbf{u}). \qquad (\dagger)$$

(ii) Deduce that $\nabla \cdot \mathbf{u}$ and $\nabla \wedge \mathbf{u}$ satisfy the wave equation with wave speeds c_p and c_s, respectively, where

$$c_p^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_s^2 = \frac{\mu}{\rho}.$$

(b) Consider now waves propagating in a two-dimensional elastic half-space with

$$-\infty < x < \infty \quad \text{and} \quad -\infty < y < 0.$$

(i) Write $\mathbf{u} = \nabla\varphi + \nabla \wedge \boldsymbol{\psi}$ and show that if the potentials φ and $\boldsymbol{\psi}$ satisfy the wave equation with wavespeeds c_p and c_s, respectively, then Navier's equation ($\dagger$) is satisfied.

(ii) Let $\mathbf{e}_z$ denote the unit vector in the direction perpendicular to the x- and y-axes. Find separable solutions to these wave equations of the form $\varphi = f(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ and $\boldsymbol{\psi} = g(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\mathbf{e}_z$ that satisfy $\mathbf{u} \to \mathbf{0}$ as $y \to -\infty$ under the assumption that $k^2 - \omega^2/c_s^2 > 0$.

(iii) Show that if the boundary condition at $y = 0$ is that $\sigma_{xy} = \sigma_{yy} = 0$ (the surface is free) then the waves satisfy the dispersion relation

$$k^2\left(k^2 - \omega^2/c_p^2\right)^{1/2}\left(k^2 - \omega^2/c_s^2\right)^{1/2} = \left(k^2 - \omega^2/2c_s^2\right)^2.$$

(iv) Are these waves dispersive or non-dispersive?

[*In this question, you may use the identity* $\nabla \wedge (\nabla \wedge \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ *for any vector field* $\mathbf{A}$.]

2012

2. (a) Let

$$I(t) = \int_a^b f(k) \exp\left[\mathrm{i}\psi(k)t\right] \, \mathrm{d}k$$

where $f(k)$ is continuously differentiable, $\psi(k)$ is real-valued and twice continuously differentiable on $[a, b]$. Explain schematically why you would expect the behaviour of $I(t)$ as $t \to \infty$ to be dominated by any points $k = k_*$ satisfying $\psi'(k_*) = 0$.

(b) Steady, small-amplitude waves disturb a fluid of constant density ρ occupying the half-space $z < 0$ and flowing with uniform velocity $U\mathbf{e}_x$.

(i) Show that the velocity potential describing the disturbance, $\phi(x, y, z)$, and the position of the disturbed free surface $z = \eta(x, y)$ satisfy

$$\begin{aligned} &\nabla^2\phi = 0 && z < 0, \\ &\frac{\partial \phi}{\partial z} = U\frac{\partial \eta}{\partial x}, \quad U\frac{\partial \phi}{\partial x} + g\eta = 0 && z = 0, \\ &\boldsymbol{\nabla}\phi \to \mathbf{0} && z \to -\infty. \end{aligned}$$

[*You may assume that the flow is potential and you may quote the non-linear Bernoulli equation.*]

(ii) Show that separable solutions with $\eta = A\mathrm{e}^{\mathrm{i}(kx+\ell y)}$ exist provided that

$$k^2 = \frac{g^2}{2U^4}\left(1 + \sqrt{1 + \frac{4U^4\ell^2}{g^2}}\right). \qquad (\star)$$

(c) An obstacle is placed along the y-axis of the flow described in (b) such that

$$\eta(0, y) = \eta_0(y) \quad \text{and} \quad \frac{\partial \eta}{\partial x} = 0 \quad \text{at} \quad x = 0.$$

(i) Show that

$$\eta(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\eta_0}(\ell)\mathrm{e}^{\mathrm{i}\ell y} \cos\big[k(\ell)x\big] \, \mathrm{d}\ell,$$

in $x > 0$, where $k(\ell)$ is given by $(\star)$ and $\widehat{\eta_0}$ is the Fourier transform of η_0, which you should define.

(ii) If $y = \lambda x$, where x is large, show that the main contribution to η arises from values of ℓ such that $\lambda = \pm \mathrm{d}k/\mathrm{d}\ell$. Hence show that in the far-field λ must satisfy

$$|\lambda| = \sqrt{\frac{s-1}{2s^2}},$$

where $s = \sqrt{1 + 4U^4\ell^2/g^2}$.

(iii) Deduce that the far-field wave pattern created by the obstacle is restricted to a wedge of internal angle $\theta = 2\sin^{-1}(1/3)$.

2012

3. (a) Consider a homentropic gas, so that $p = S\rho^\gamma$ for some positive constants S and γ.

(i) Show that the equations of one-dimensional gas dynamics

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

can be written as

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)\left(u \pm \frac{2c}{\gamma - 1}\right) = 0,$$

where $c^2 = \gamma p/\rho$.

(ii) What is the significance of the Riemann invariants $R_\pm = u \pm 2c/(\gamma - 1)$?

(b) Consider a one-dimensional flow in which homentropic gas is initially stationary and confined to $x \in [0, \infty)$. Let $c = c_0$, a constant, denote the sound speed. At $t = 0$ the boundary at $x = 0$ begins to move according to $x = -at^2$.

(i) Show that subsequently $u = 0$, $c = c_0$ for $x > c_0 t$.

(ii) Show that for $x < c_0 t$ and $t < c_0/[a(\gamma - 1)]$

$$u = -2a\tau, \quad c = c_0 - a(\gamma - 1)\tau,$$

where τ satisfies

$$x = -a\tau^2 + [c_0 - (\gamma + 1)a\tau](t - \tau).$$

(c) Consider now a function $v(x,t)$ satisfying the equation

$$\frac{\partial v}{\partial t} + (\alpha + \beta v)\frac{\partial v}{\partial x} = -\delta v$$

with initial condition $v(x, 0) = v_0(x)$ and α, β, δ positive constants.

(i) Show that $v(x,t) = v_0(x_0)\mathrm{e}^{-\delta t}$ where $x_0(x,t)$ satisfies

$$x = \alpha t + x_0 + \beta v_0(x_0)\frac{1 - \mathrm{e}^{-\delta t}}{\delta}.$$

(ii) Explain why no shock can form if

$$\delta > \beta \max_{v_0'(x_0) < 0} \left\{|v_0'(x_0)|\right\}.$$

- 2012 Q1(b)(iv) Could you talk through whether the waves are dispersive?

$$k^2\left(k^2-\omega^2/c_p^2\right)^{1/2}\left(k^2-\omega^2/c_s^2\right)^{1/2}=\left(k^2-\omega^2/2c_s^2\right)^2.$$

Phase speed $c = \omega/k$ s.t. $\left(1-\frac{c^2}{c_p^2}\right)^{1/2}\left(1-\frac{c^2}{c_s^2}\right)^{1/2} = \left(1-\frac{c^2}{2c_s^2}\right)^2$

$\Rightarrow$ c ind. k $\Rightarrow$ non-dispersive.

- 2012 Q2(b) Could you just talk through how to apply FT for this question?

From 2017 paper hints on course website.

Note $\hat{\phi}(x,l,z) = \int_{-\infty}^{\infty} \phi e^{-ily}\,dy$, $\hat{m}(x,l) = \int_{-\infty}^{\infty} m e^{-ily}\,dy$

$\Rightarrow$ $\hat{\phi}_{xx} + \hat{\phi}_{zz} = l^2\hat{\phi}$ in $z<0$ with $\hat{\phi}_z = U\hat{m}_x$
and $U\hat{\phi}_x + g\hat{m} = 0$ on $z=0$; $\hat{\phi}_x \to 0$ as $z\to-\infty$
and $\hat{m} = \hat{m}_0$, $\hat{m}_x = 0$ at $x=0$.

Key step: seek sep. soln $\hat{\phi} = X(x)Z(z)$.

$\Rightarrow$ $\frac{X''}{X} + \frac{Z''}{Z} = l^2$ $\Rightarrow$ $\frac{X''}{X} = \text{const.} = -k^2$

for oscillatory solutions in the x-direction.

$\Rightarrow$ $\hat{\phi} = \left(A(l)\cos kx + B(l)\sin kx\right)e^{(k^2+l^2)^{1/2}z}$

$[\hat{m} = -\frac{U}{g}\hat{\phi}_x \text{ on } z=0]$ $\quad$ $[g\hat{\phi}_z = -U^2\hat{\phi}_{xx} \text{ on } z=0]$

BCs on $z=0$ $\Rightarrow$ $\hat{m} = \frac{kU}{g}(A\sin kx - B\cos kx)$, $(k^2+l^2)^{\frac{1}{2}} = \frac{U^2k^2}{g}$

ICs on $x=0$ $\Rightarrow$ $A=0$, $\hat{m}_0 = -\frac{BkU}{g}$ $\Rightarrow$ $\hat{m} = \hat{m}_0\cos(kx)$
and invert.

- ~~2021~~ 2012 Q3(c)(ii) Unsure of how to approach this part!!

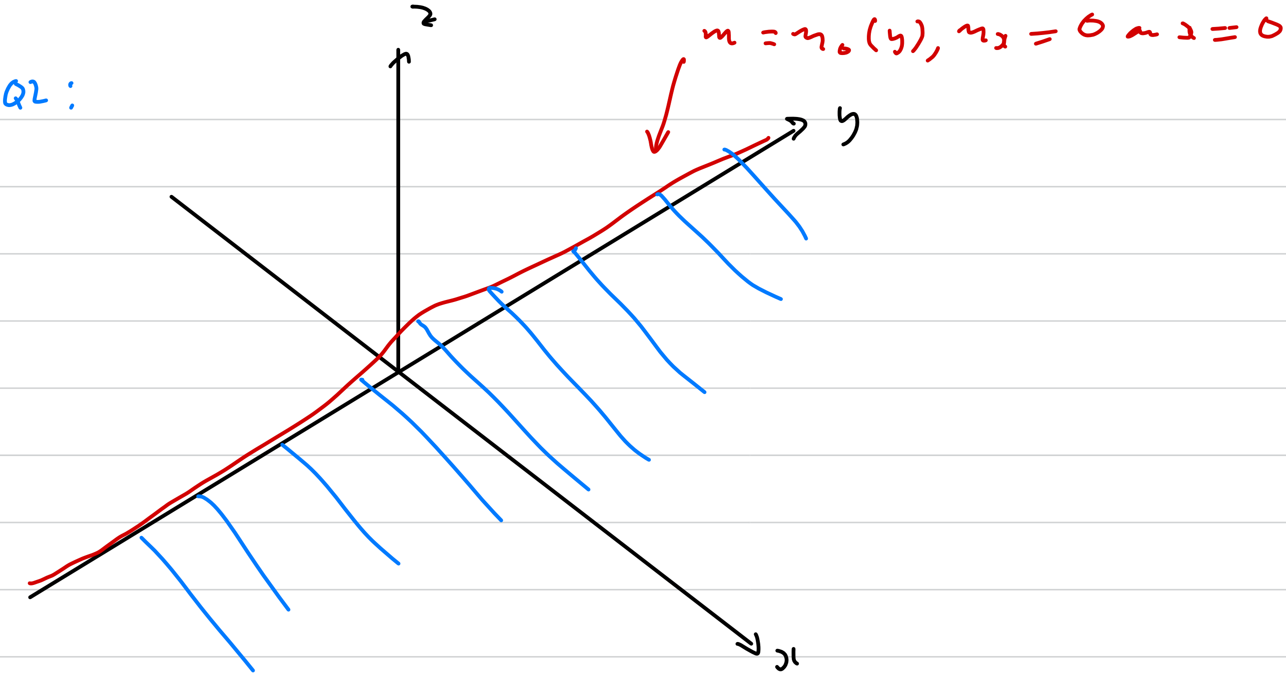
Part A DEs I $\Rightarrow$ $\frac{dv}{dt} = -\delta v$ on $\frac{dx}{dt} = \alpha + \beta v$ with $v = v_0(x)$, $x = x_0$ at $t=0$

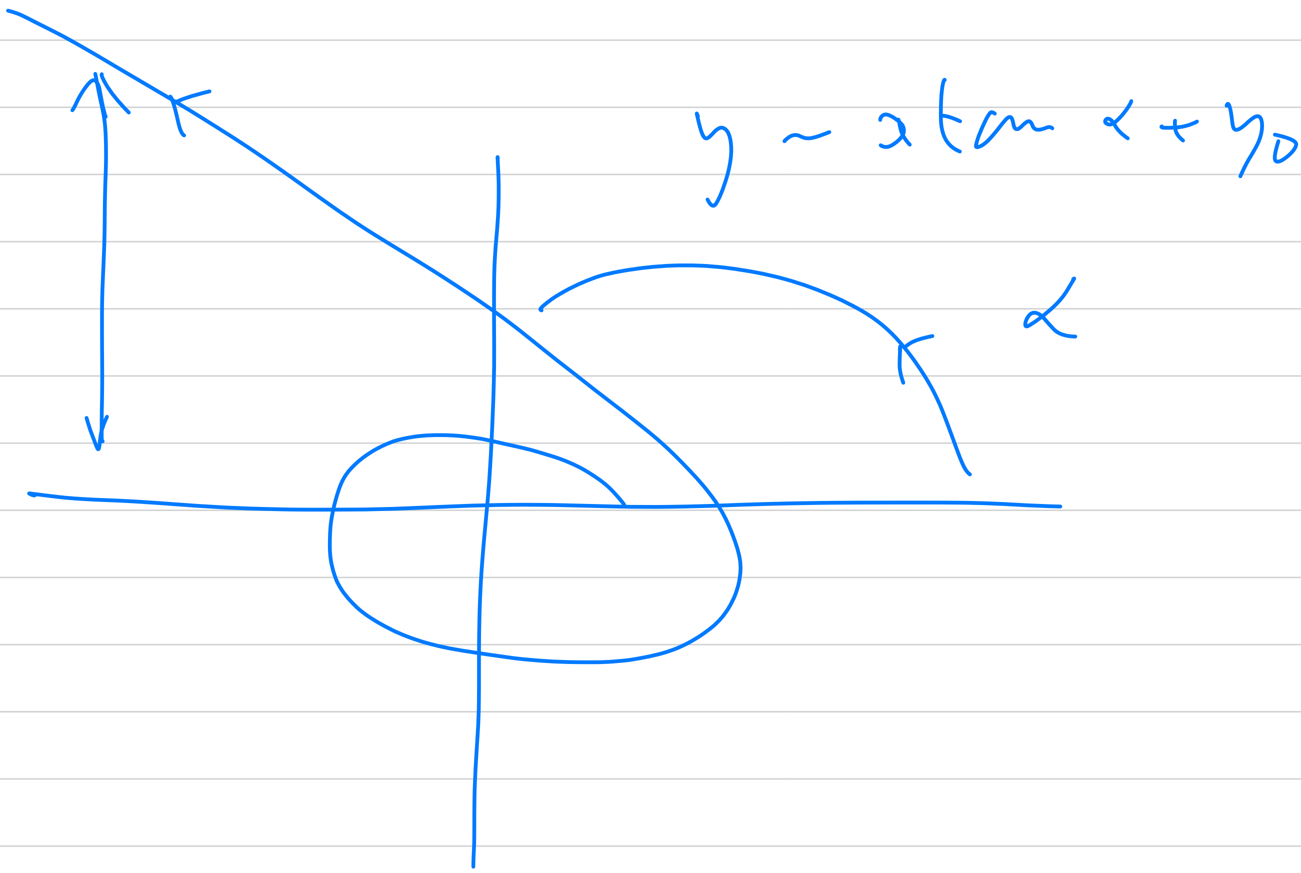
$\Rightarrow$ $v = v_0(x_0)e^{-\delta t}$ & $\frac{dx}{dt} = \alpha + \beta v_0(x_0)e^{-\delta t}$

$\Rightarrow$ $x = x_0 + \alpha t + \beta v_0(x_0)\frac{1-e^{-\delta t}}{\delta}$ $\quad$ $[\Rightarrow x_0 = x_0(x,t)]$

Hence, $v_x = e^{-\delta t}v_0'(x_0)\frac{\partial x_0}{\partial x}$, where $1 = \left(1 + \beta v_0'(x_0)\frac{1-e^{-\delta t}}{\delta}\right)\frac{\partial x_0}{\partial x}$ $\square$

Figures for Q2:





$v_x = e^{-\delta t} v_0'(x_0)\frac{\partial x_0}{\partial x}$, where $1 = \left(1 + \beta v_0'(x_0)\frac{1-e^{-\delta t}}{\delta}\right)\frac{\partial x_0}{\partial x}$

$$\Rightarrow \quad v_x = \frac{\delta e^{-\delta t} v_0'(x_0)}{\delta + \beta v_0'(x_0)\left(1 - e^{-\delta t}\right)}$$

Shock forms if $\exists x_0, t$ s.t. $\delta = -\beta v_0'(x_0)(1-e^{-\delta t})$ $(*)$

Since $\delta, \beta > 0$, this can only be the case if $v_0'(x_0) < 0$, in which case $\mathrm{RHS}(*) < -\beta v_0'(x_0)$ for $t \geq 0$.

Hence if $\delta > \beta \max\limits_{v_0'(x_0)<0} |v_0'(x_0)|$, then $\delta > \mathrm{RHS}(*)$ $\forall x_0, t$, so that a shock cannot form.

NB: there are alternative equivalent methods (intersecting characteristics or vanishing Jacobian).

NB: This is more DEs I / BS.2 than BS.4.

2013

1. A barotropic gas with pressure–density relation $p = P(\rho)$ is initially stationary and has uniform density ρ_0.

(a) Starting from the Euler equations for an inviscid compressible fluid, show that small amplitude irrotational perturbations to the initial state are governed by

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi$$

where ϕ is a potential for the velocity perturbations, and c_0^2 is a constant whose value you should state clearly. Write down expressions for the pressure and density perturbations p' and ρ' in terms of ϕ.

(b) The gas is confined to lie in a one-dimensional, semi-infinite tube $X(t) < x < \infty$. The motion of the end of the tube is prescribed so that $X(t) = a\mathrm{e}^{-\mathrm{i}\omega t}$ where a is small and the real part is understood.

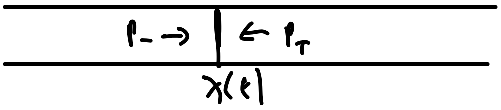
By linearising the appropriate boundary condition at $x = X(t)$ onto $x = 0$, show that

$\phi_x = \dot{X} = -i\omega a e^{-i\omega t}$ on $x = a e^{-i\omega t}$

$\Rightarrow \phi_x = -i\omega a e^{-i\omega t}$ on $x = 0$ linearizing.

$$\phi = -ac_0 \exp\left[\mathrm{i}\omega\left(\frac{x}{c_0} - t\right)\right].$$

(c) The gas is now confined to lie in a one-dimensional, infinite tube $-\infty < x < \infty$. An infinitely thin barrier of mass per unit area m is introduced at $x = X(t)$ and moves in response to the difference in gas pressure between its two sides according to



$$m\frac{\mathrm{d}^2 X}{\mathrm{d}t^2} = p|_{x=X^-} - p|_{x=X^+}\,. \qquad (\dagger)$$

A travelling wave $\phi = \exp\left[\mathrm{i}\omega\left(\frac{x}{c_0} - t\right)\right]$ is incident on the left-hand side of the barrier.

(i) Explain briefly why the appropriate solution of the wave equation may be written in the form

$$\phi(x,t) = \begin{cases} \exp\left[\mathrm{i}\omega\left(\frac{x}{c_0} - t\right)\right] + R\exp\left[-\mathrm{i}\omega\left(\frac{x}{c_0} + t\right)\right], & x < X(t), \\ T\exp\left[\mathrm{i}\omega\left(\frac{x}{c_0} - t\right)\right], & x > X(t), \end{cases}$$

for some constants R and T.

(ii) By assuming small deflections of the barrier away from the origin of the form $X = x_0 e^{-\mathrm{i}\omega t}$ solve for the constants x_0, R and T.

(iii) Find the behaviour of R and T in the limits $\omega \to 0$ and $\omega \to \infty$. Why do the low frequency notes of a noisy neighbour's music penetrate a wall more effectively than do the high frequency notes?

2013

2. Consider the steady flow of a barotropic gas with pressure–density relation $p = P(\rho)$. In the absence of any disturbance, the flow is uniform with speed U in the $x-$direction, density ρ_0 and pressure $p_0 = P(\rho_0)$. You may assume that once disturbed the flow remains irrotational, so that the velocity is given by $\mathbf{u} = U\mathbf{i} + \nabla\phi$, where $\nabla\phi \ll U$.

(a) (i) Starting from the Euler equations for an inviscid compressible fluid, find an expression for the pressure in terms of ϕ and show that steady two-dimensional disturbances are governed by the equation

$$0 = (1 - M^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \qquad (\ddagger)$$

where $M = U/c_0$ and c_0 is the speed of sound, which you should define.

(ii) Describe, briefly how the mathematical character of ($\ddagger$) is different depending on whether $M < 1$ or $M > 1$. When $M > 1$, what is the significance of the lines $x \pm (M^2 - 1)^{1/2}y = \text{constant}$?

(iii) A rigid wing is placed along $-a \leqslant x \leqslant a$, $y = \pm f(x)$ with $f(x) \ll a$. Indicate with a sketch the regions in which the gas is undisturbed by the presence of the wing and those regions in which the gas is disturbed.

(b) The flow described above with $M < 1$ occupies the semi-infinite space, $y \geqslant 0$. A pressure perturbation $p' = -\rho_0 U f(x)$, for some function $f(x)$, is applied at $y = 0$.

(i) What is the boundary condition for ϕ on $y = 0$?

(ii) Use a Fourier transform in x to find the velocity potential $\phi(x, y)$.

[*You may assume that the Fourier transform of* $\tan^{-1}(x/\epsilon)$ *is* $-\pi \mathrm{i} e^{-\epsilon|k|}/k$ *as well as the convolution theorem that if* $\widehat{g}(k) = \widehat{h}_1(k)\widehat{h}_2(k)$ *then*

$$g(x) = \int_{-\infty}^{\infty} h_1(\xi)h_2(x - \xi)\ \mathrm{d}\xi.]$$

(c) Steady disturbances to a weakly stratified gas satisfy ($\ddagger$) with $M = (1 + y)^{1/2}$.

(i) How does the mathematical character of ($\ddagger$) depend on whether $y > 0$ or $y < 0$ in this case?

(ii) Consider a wave of the form $\phi = \mathrm{Re}[e^{ikx}A(y)]$ incident from above on $y > 0$. Deive a differential equation for $A(y)$ and explain the behaviour of the solution in the regions $y > 0$ and $y < 0$ with reference to your answer to (c)i.

[*You may assume that the relevant solution of Airy's equation*

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} - \xi\eta = 0$$

is oscillatory in $\xi < 0$ *but decays exponentially in* $\xi > 0$.]

- 2013 Q1(c) Could you talk through the BCs we need to apply here?

(ii) KBC $\Rightarrow \quad \phi_x = \dot{x}$ on $x = X\pm$ $\underset{x = x_0 e^{-i\omega t}}{\Rightarrow}$ $\phi_x = -i\omega x_0 e^{-i\omega t}$ on $x = 0\pm$ linearizing

N II $\Rightarrow \quad m\ddot{x} = [p']^{x=X-}_{x=X+} = [\rho_0 \phi_t]^{x=X+}_{x=X-} \Rightarrow m(-i\omega)^2 x_0 e^{-i\omega t} = [\rho_0 \phi_t]^{x=0+}_{x=0-}$ linearizing

Three BCs give three equations for three unknowns : R, T, x_0

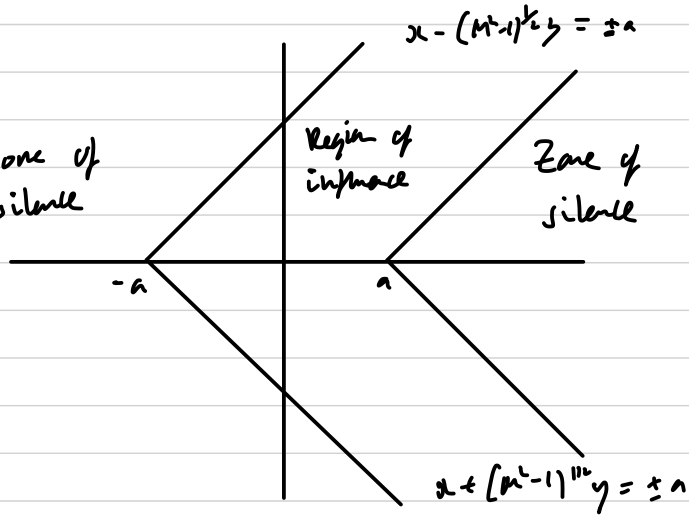
Sub. for ϕ into linearized BC $\Rightarrow \quad x_0 = -\dfrac{2i\rho_0}{m\omega + 2i\rho_0 c_0}, \quad T = \dfrac{2i\rho_0 c_0}{m\omega + 2i\rho_0 c_0}, \quad R = \dfrac{m\omega}{m\omega + 2i\rho_0 c_0}$

(iii) Hence, $R \to 0, \; T \to 1$ as $\omega \to 0$ (low frequency)
$R \to 1, \; T \to 0$ as $\omega \to \infty$ (high frequency)

This means low frequencies penetrate walls more efficiently and high frequencies reflected.

- 2013 Q2(a)(iii) Could you just quickly cover region of silence?
- 2013 Q2(b) I think I got the correct BCs and solution, but could I perhaps check these?
- 2013 Q2(c)(ii) Less sure on this part. Behaviour changes at y=0 but not sure what this means!

(a)(iii)



[Information travels along char's $x \pm (M^2-1)^{1/2} y =$ const. from left to right by causality. Those char's that come from $x = -\infty$ without intersecting wing $[-a, a] \times \{0\}$ carry $\nabla\phi = \underline{0}$, i.e. zones of silence. Those that intersect the wing carry information about wing, i.e. region of influence.]

(b)(i) Have $(1 - M^2)\phi_{xx} + \phi_{yy} = 0$ in $y > 0$

with $-\rho_0 U \phi_x = p' = -\rho_0 U f(x) \Rightarrow \phi_x = f(x)$ on $y = 0$

and $\nabla\phi \to \underline{0}$ as $y \to \infty$.

(ii) $\hat{\phi}(k, y) = \int_{-\infty}^{\infty} \phi(x, y) e^{-ikx} dx \Rightarrow (ik)^2 (1 - M^2)\hat{\phi} + \hat{\phi}_{yy} = 0$ for $y > 0$

with $(ik)\hat{\phi} = \hat{f}$ on $y = 0$ and $\hat{\phi} \to 0$ as $y \to \infty$

Hence, $\hat{\phi} = \dfrac{\hat{f}(k)}{ik} e^{-(1-M^2)^{1/2}|k| y} = \hat{f}(k)\hat{g}(k)$, $\quad g(x) = \dfrac{1}{\pi} \tan^{-1}\left(\dfrac{x}{\sqrt{1-M^2}\, y}\right)$ by hint, so convolution then gives $\phi(x,y) = \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi) \tan^{-1}\left(\frac{x-\xi}{\sqrt{1-M^2}\,y}\right) d\xi$

(c)(i) $M = (1+y)^{1/2} \Rightarrow y\phi_{xx} = \phi_{yy}$, which is hyperpolic if $y>0$
elliptic if $y<0$.

(c)(ii) $\phi = e^{ikx} A(y)$ (RPU) $\Rightarrow -k^2 y A = \frac{d^2A}{dy^2} \Rightarrow \frac{d^2A}{d\xi^2} - \xi A = 0$, $\xi = -k^{2/3} y$

A is an Airy function so solⁿ oscillatory for $y>0$ but decays exponentially for $y<0$,

This fits behaviour in (c)(i) because disturbances should propagate when PDE is hyperbolic but decay when it is elliptic. [Think of far-field decay in (b) or in Stokes waves.]

2016

3. (a) [6 marks] The shallow-water equations are

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0,$$

where $h(x,t)$ is the fluid depth and $u(x,t)$ is its velocity. Show that these equations can be rewritten as

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)(u \pm 2c) = 0,$$

where $c = \sqrt{gh}$. State the significance of the quantities $u \pm 2c$ and the curves $\mathrm{d}x/\,\mathrm{d}t = u \pm c$.

(b) [8 marks] A partition at $x = 0$ separates stationary fluid with depth h_L in $x < 0$ from stationary fluid with depth $h_R < h_L$ in $x > 0$. At $t = 0$, the partition is removed.

The flow separates into a stationary region $x < at$, an expansion fan in $at < x < bt$, a region of uniform depth $bt < x < Vt$, and a stationary region $x > Vt$, where $a < 0$, b, and $V > 0$ are constants.

With reference to characteristic curves, explain why the fluid depths and velocities immediately to the left $(-)$ and right $(+)$ of the shock at $x = Vt$ must satisfy

$$u_- + 2\sqrt{gh_-} = 2\sqrt{gh_L}, \qquad u_+ + 2\sqrt{gh_+} = 2\sqrt{gh_R},$$

and show further that $h_+ = h_R$. Using the shock conditions, show that the shock speed V and upstream depth h_- are related by

$$V = \sqrt{\frac{g}{2}\frac{(h_- + h_R)h_-}{h_R}} = \frac{2(\sqrt{gh_L} - \sqrt{gh_-})h_-}{h_- - h_R}.$$

Hence show that the depth h_-, and consequently u_- and V, are uniquely determined. [*Do not try to solve for them explicitly.*]

(c) [8 marks] Noting that negative characteristics, on which $u - 2\sqrt{gh}$ is constant, emanate from behind the shock, solve for the flow in each of the regions upstream of the shock, giving expressions for the constants a and b in terms of h_L and h_-.

(d) [3 marks] Sketch the water depth as a function of x at $t = t_* > 0$.

[*You may assume the conditions for a shallow water shock moving with speed* V*,*

$$[h(u-V)]_-^+ = \left[h(u-V)^2 + \tfrac{1}{2}gh^2\right]_-^+ = 0,$$

where $[\ ]_-^+$ *denotes the change in the quantity from one side of the shock to the other.*]

- 2016 Q3(b) Why must the fluid depths and velocities satisfy these relations to the left and the right of x = Vt? How come the shock / expansion fan doesn't at x = bt doesn't get in the way? Why do we know that the shock is a straight line?

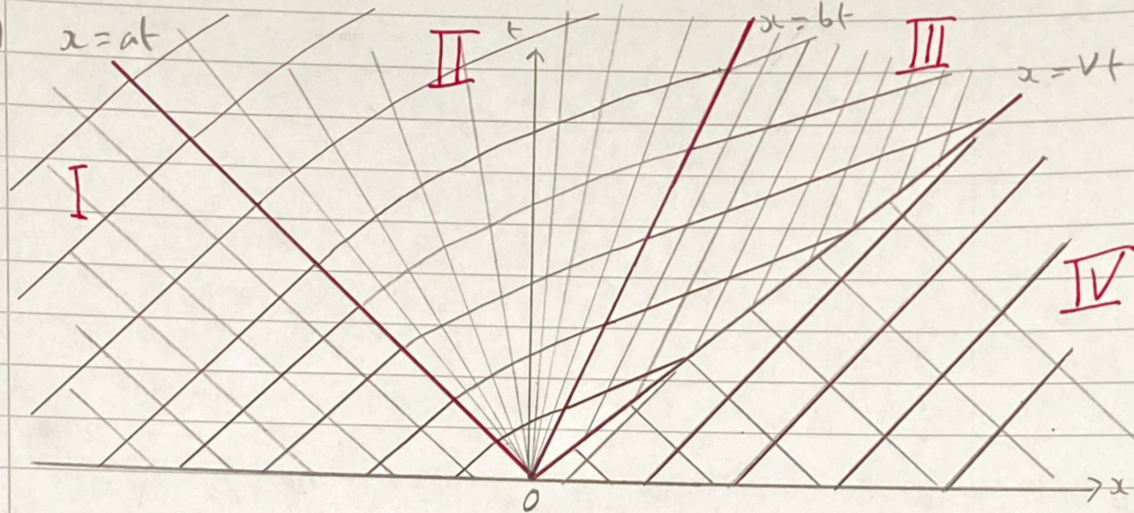
See hint for 2016 paper below that describes the flow of information into and out of the shock.

That the shock and boundaries between regions are straight may be anticipated by the problem being invariant to the linear scaling $(x,t) \mapsto \lambda(x,t)$ $\forall_{\lambda>0}$, which implies that h and u are functions only of x/t.

B5.4 | 2016 | Q3

(a) B - see online notes §4.3 & sheet 4, Q1

(b-c)



Region	C_- char^{cs}	C_+ char^{cs}
I	From $\{x<0, t=0\}$, so $u-2c=-2c_L$	From $\{x<0, t=0\}$, so $u+2c=2c_L$.
II	From $\{x=0, t=0\}$, with $u-c=\frac{x}{t}$	
III	From $\{x=Vt, t>0\}$, with $u-2c=u_--2c_-$	
IV	From $\{x>0, t=0\}$, so $u-2c=-2c_R$	From $\{x>0, t=0\}$, so $u+2c=2c_R$

NB: C_- char^{cs} everywhere straight; C_+ char^{cs} straight except in expansion fan region II.

C_+ char^{cs} carry info into LHS shock $\Rightarrow u_- + 2c_- = 2c_L$

$C_\pm$ char^{cs} carry info into RHS shock $\Rightarrow u_+ = 0,\ c_+ = c_R$

$$\text{RHCs} \underset{(u_+=0)}{\Rightarrow} V = -\frac{h_-u_-}{h_+-h_-} = \left(\frac{g(h_++h_-)h_-}{2h_+}\right)^{1/2}$$

$\rightarrow$ combo to get expressions for V in part (b); sketch e.g. $V(h_- - h_R)$ as a f^n of h_- two ways to get uniqueness of h_- & hence u_- & V.

Into above $\Rightarrow$ $a = -c_L$, $b = 2c_L - 3c_-$; $c = c_L$, $u = 0$ in I; $c = \frac{1}{3}\left(2c_L - \frac{x}{t}\right)$, $u = \frac{2}{3}\left(c_L + \frac{x}{t}\right)$ in II; $c = c_-$, $u = 2(c_L - c_-)$ in III; and $c = c_R$, $u = 0$ in IV.

2017

2. (a) [4 marks] Let

$$I(x) = \int_{-\infty}^{\infty} f(\ell) e^{i\psi(\ell)x} \, \mathrm{d}\ell,$$

where $f(\ell)$ is an integrable function and $\psi(\ell)$ is twice continuously differentiable. Explain schematically why you would expect the behaviour of $I(x)$ as $x \to \infty$ to be dominated by values of $\ell = \ell_*$ for which $\psi'(\ell_*) = 0$.

(b) [10 marks] Steady small-amplitude waves disturb a fluid of constant density ρ that occupies the region $z < \eta(x, y)$ and moves with constant background velocity $U\hat{\mathbf{e}}_x$, where $\hat{\mathbf{e}}_x$ is the unit vector in the x direction. The disturbances satisfy the linearised equations

$$\nabla^2 \phi = 0, \qquad z < 0,$$

$$\frac{\partial \phi}{\partial z} = U \frac{\partial \eta}{\partial x}, \quad U \frac{\partial \phi}{\partial x} + g\eta = 0 \qquad \text{on} \quad z = 0,$$

$$\frac{\partial \phi}{\partial z} \to 0 \qquad \text{as} \quad z \to -\infty.$$

An obstacle placed on the y-axis provides the conditions

$$\eta = \eta_0(y), \quad \frac{\partial \eta}{\partial x} = 0 \qquad \text{at} \quad x = 0.$$

By taking a Fourier transform in y, or otherwise, show that

$$\eta(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\eta}_0(\ell) e^{i\ell y} \cos\left(k(\ell)x\right) \, \mathrm{d}\ell,$$

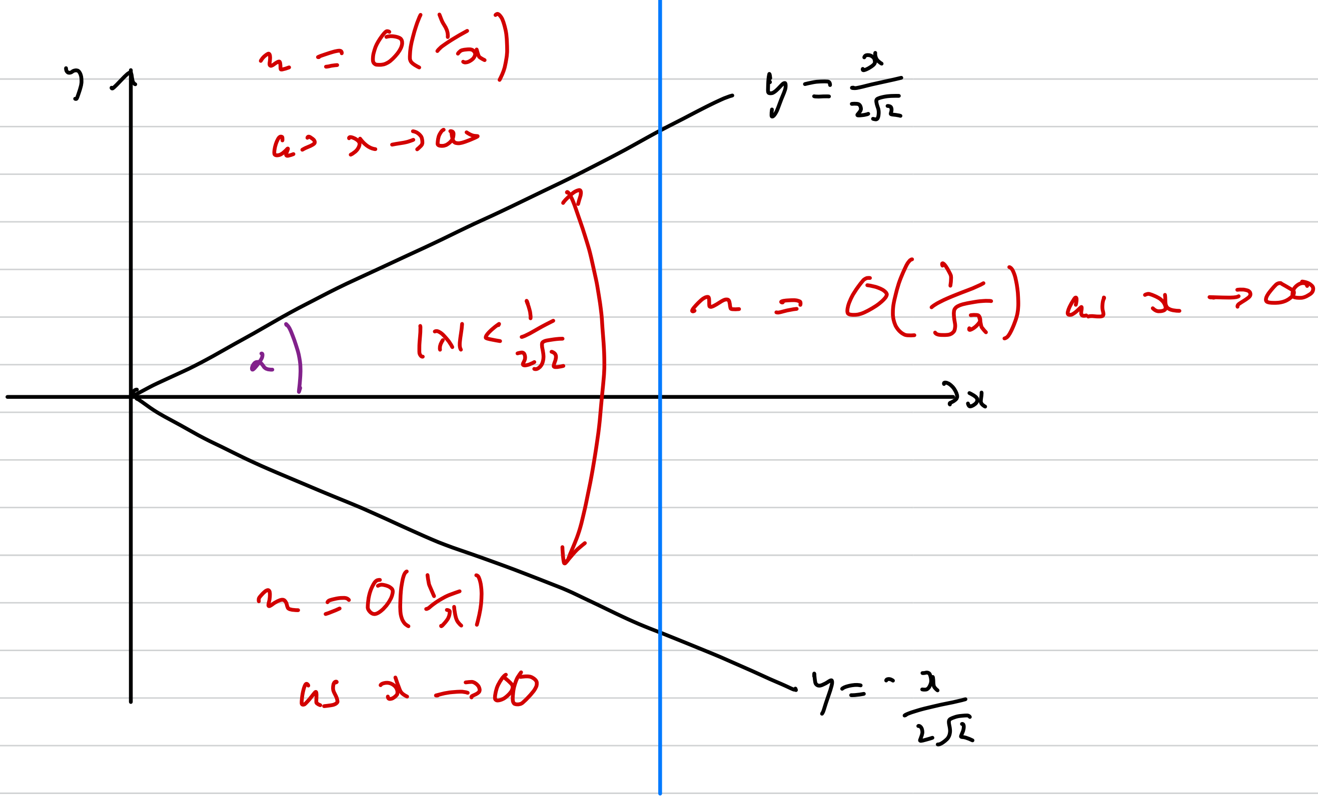
where $\hat{\eta}_0(\ell)$ is the Fourier transform of $\eta_0(y)$, and where

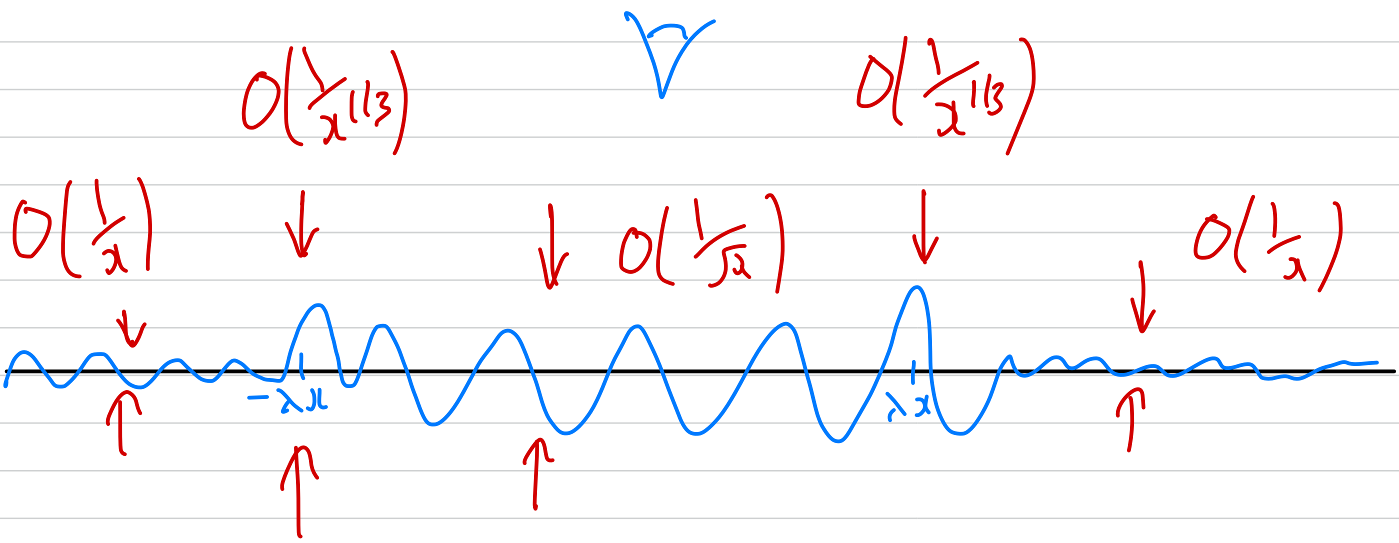
$$k(\ell) = \sqrt{\frac{g^2(1+s)}{2U^4}}, \qquad s(\ell) = \sqrt{1 + 4U^4\ell^2/g^2}.$$

(c) [7 marks] By considering $y = \lambda x$ for large x, and making use of (a), show that the dominant contribution to the wave pattern arises from values of ℓ such that $\lambda = \pm \mathrm{d}k/\mathrm{d}\ell = \pm\sqrt{(s-1)/2s^2}$.

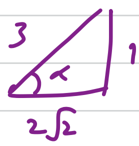
Hence show that the dominant wake of the obstacle is confined to a wedge, the edges of which make an angle $\sin^{-1}(1/3)$ with the x-axis.

(d) [4 marks] Show further that the crests of the waves that are seen at the edge of the wake are aligned at an angle $\cos^{-1}(1/\sqrt{3})$ with the x-axis, and that these waves have wavelength $4\pi U^2/3g$.





$\tan\alpha = \frac{1}{2\sqrt{2}} \iff \sin\alpha = \frac{1}{3}$



Q2(a)

$$\eta(x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{\eta}_0(l)\left(e^{i(kx+ly)} + e^{i(-kx+ly)}\right) dl$$

Superposition of waves with wavevector $\underline{k} = (k, l)$

B5.4/2017/Q2

(a) B – see online notes §3.5.

(b,c) B/S – see sheet 3, Q1.

Note $\hat{\phi}(x,l,z) = \int_{-\infty}^{\infty} \phi e^{-ily}\,dy$, $\hat{n}(x,l) = \int_{-\infty}^{\infty} n e^{-ily}\,dy$

$\Rightarrow \hat{\phi}_{xx} + \hat{\phi}_{zz} = l^2\hat{\phi}$ in $z<0$ with $\hat{\phi}_z = U\hat{n}_x$
and $U\hat{\phi}_x + g\hat{n} = 0$ on $z=0$; $\hat{\phi}_z \to 0$ as $z\to-\infty$
and $\hat{n} = \hat{n}_0$, $\hat{n}_x = 0$ at $x=0$.

Key step: seek sep. soln $\hat{\phi} = X(x)Z(z)$.

$\Rightarrow \frac{X''}{X} + \frac{Z''}{Z} = l^2 \Rightarrow \frac{X''}{X} = \text{const.} = -k^2$

for oscillatory solutions in the x-direction.

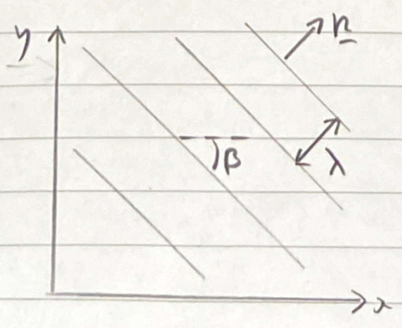
$\Rightarrow \hat{\phi} = (A(l)\cos kx + B(l)\sin kx)\,e^{(k^2+l^2)^{1/2}z}$

BCs on $z=0 \Rightarrow \hat{n} = \frac{kU}{g}(A\sin kx - B\cos kx)$, $(k^2+l^2)^{\frac{1}{2}} = \frac{U^2k^2}{g}$

ICs on $x=0 \Rightarrow A=0$, $\hat{n}_0 = -\frac{BkU}{g} \Rightarrow \hat{n} = \hat{n}_0\cos(kx)$
and invert.

(d) Waves at edge of wake ($\lambda = \frac{1}{\sqrt{8}}$ say) are those with $s=2$, i.e. $l = \mp\frac{g}{U^2}\frac{\sqrt{3}}{2}$, $k = \frac{g}{U^2}\sqrt{\frac{3}{2}}$.

Hence, wavenumber vector $\underline{k} = (k,l) = \frac{g}{U^2}\sqrt{\frac{3}{2}}\left(1, \mp\frac{1}{\sqrt{2}}\right)$



Wavecrests are $\underline{k}\cdot(x,y) = \text{const}$ as illustrated for + sign.

$\Rightarrow \cos\beta = \frac{1}{\sqrt{3}}$, $\lambda = \frac{2\pi}{|\underline{k}|} = \frac{4\pi U^2}{3g}$