B5.4 Waves & Compressible Flow **Trinity Term**

Consultation session questions

- A quick question about supersonic flow past a thin wing. On page 60 of the notes, where we're finding the value of ϕ in each of the 6 regions, in regions 3 and 6, is ϕ supposed to be 0 or a constant (so that we have continuity between regions 2 and 3, 5 and 6)? The notes say that ϕ should be 0 (so that they are zones of silence) but in the lectures, you imposed continuity so that ϕ are non-zero constants instead, so I was just wondering which one I should use? Can we still say that there are zones of silence if ϕ is non-zero?
- 2012 Q1(b)(iv) Could you talk through whether the waves are dispersive?
- 2012 Q2(b) Could you just talk through how to apply FT for this question?
- 2021 Q3(c)(ii) Unsure of how to approach this part!!
- 2013 Q1(c) Could you talk through the BCs we need to apply here?
- 2013 Q2(a)(iii) Could you just quickly cover region of silence?
- 2013 Q2(b) I think I got the correct BCs and solution, but could I perhaps check these?
- 2013 Q2(c)(ii) Less sure on this part. Behaviour changes at y=0 but not sure what this means!
- 2016 Q3(b) Why must the fluid depths and velocities satisfy these relations to the left and the right of x = Vt? How come the shock / expansion fan doesn't at x = bt doesn't get in the way? Why do we know that the shock is a straight line?
- 2017 Q2(c,d) Please could you go over them.
- 2017 Q3(c) Please could you go over it.
- 2014 Q1(c) We will end up with $AJ_0 + BY_0$ as our solution, but could you just discuss the BCs we need here?
- 2014 Q3(d) Not quite sure how to combine previous parts? Was happy with all previous parts however.
- 2017 Q1(c) I'd much appreciate if we could go through the computations.

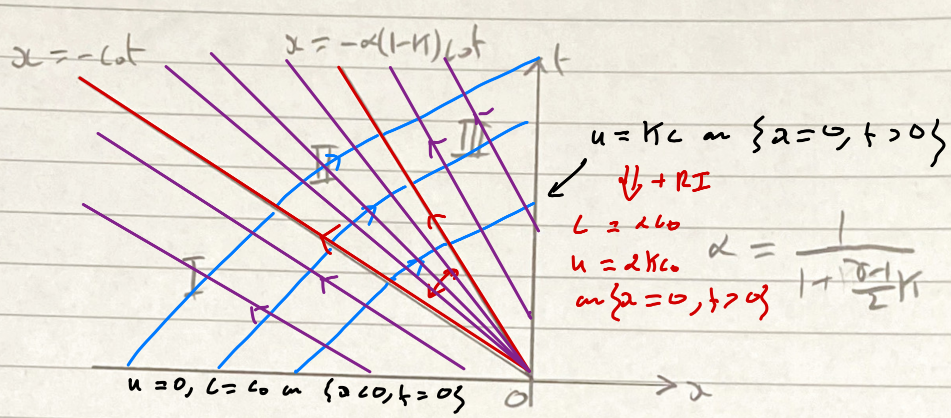
2017 Q3(c) Please could you go over it.

B5.4/2017/Q3

(a) B – see online notes §1.9 and sheet 0, Q3.

(b) B – see online notes §4.2 & sheet 3, Q3

(c)



C_+ char^{cs} on which $u+2c=2c_0$ come everywhere from $\{x<0, t=0\}$.

I : C_- char^{cs} on which $u-2c=-2c_0$ come from $\{x<0, t=0\}$ $\Rightarrow$ $u=0$, $c=c_0$.

III : C_- char^{cs} on which $u-2c=\text{const}$ come from $\{x=0, t>0\}$ on which $u=kc$ $\Rightarrow$ $c=\alpha c_0$, $u=\alpha k c_0$

II : Needed because $-c_0t < -\alpha(1-k)c_0t < 0$ for $t>0$ because $0<\alpha<1$ and $0<k<1$. Now C_- char^{cs} come from origin (expansion fan)

$$\Rightarrow \quad c = \frac{2}{\gamma+1}\left(c_0 - \frac{\gamma-1}{2}\frac{x}{t}\right), \quad u = \frac{2}{\gamma+1}\left(c_0 + \frac{x}{t}\right)$$

$u+2c=2c_0$ (from $\{x<0, t=0\}$)

$\dfrac{dx}{dt} = u-c = \dfrac{x}{t}$ (as $-$char^{cs} must originate from $\{x=0, t=0\}$)

2014

1. A barotropic gas has pressure–density relation $p = P(\rho)$ and is at rest with uniform density ρ_0 and pressure $p_0 = P(\rho_0)$ such that $P'(\rho_0) = 1$. You may assume that small amplitude perturbations to the uniform state are described by a velocity potential ϕ, which satisfies the radially symmetric wave equation

$$\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2},$$

and that the corresponding pressure perturbations are given by

$$p - p_0 = -\rho_0\frac{\partial\phi}{\partial t}.$$

A circular cylinder with radius 1 and length 1 is placed in the gas. Suppose the walls are rigid, but that the ends at $z = 0$ and $z = 1$ are open, so that the pressure there is fixed at p_0.

(a) [12 marks] Write down appropriate boundary conditions for the velocity potential inside the cylinder.

Seeking a separable solution $\phi(r, z, t) = e^{-i\omega t}f(r)g(z)$, find the natural frequencies for radially symmetric normal modes, and show that the lowest of these is given by

$$\omega^2 = \pi^2 + \xi_1^2,$$

where ξ_1 is the first positive zero of $J_0'(\xi)$.

Suppose instead that the cylinder is infinitely long, and that the walls undergo small-amplitude oscillations with radius given by $R(t) = 1 + \epsilon e^{-i\omega t}$ (the real part is assumed), perturbing the gas both inside and outside the cylinder.

(b) [6 marks] Explain why it is appropriate to prescribe the boundary condition

$$\frac{\partial\phi}{\partial r} = -i\omega\epsilon e^{-i\omega t} \quad \text{at} \quad r = 1.$$

Hence find the solution for the perturbed velocity potential $\phi(r, t)$ inside the cylinder. For which values of ω is this solution invalid?

(c) [7 marks] Show that the solution for the perturbed velocity potential $\phi(r, t)$ outside the cylinder is

$$\phi(r, t) = -i\epsilon e^{-i\omega t}\frac{J_0(\omega r) + iY_0(\omega r)}{J_0'(\omega) + iY_0'(\omega)}.$$

[*You may make use of the two linearly independent solutions, $J_0(\xi)$ and $Y_0(\xi)$, to Bessel's equation of order zero,*

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}f}{\mathrm{d}\xi} + f = 0,$$

and of their asymptotic behaviour,

$$\left.\begin{aligned} J_0(\xi) &\sim 1 - \tfrac{1}{4}\xi^2 \\ Y_0(\xi) &\sim \tfrac{2}{\pi}\ln\left(\tfrac{\xi}{2}\right) \end{aligned}\right\} \quad \text{as} \quad \xi \to 0, \qquad \left.\begin{aligned} J_0(\xi) &\sim \sqrt{\tfrac{2}{\pi\xi}}\cos(\xi - \pi/4) \\ Y_0(\xi) &\sim \sqrt{\tfrac{2}{\pi\xi}}\sin(\xi - \pi/4) \end{aligned}\right\} \quad \text{as} \quad \xi \to \infty.$$]

2014 Q1(c) We will end up with $AJ_0 + BY_0$ as our solution, but could you just discuss the BCs we need here?

KBC: $\frac{\partial \phi}{\partial r} = \underline{u} \cdot \underline{e}_r = \frac{dR}{dt}$ on $r = R(t) = 1 + \varepsilon e^{-i\omega t}$ (RPU)

By Taylor's Thm $\frac{\partial \phi}{\partial r}(1+\varepsilon e^{-i\omega t}, t) = \frac{\partial \phi}{\partial r}(1,t) + \varepsilon e^{-i\omega t} \frac{\partial^2 \phi}{\partial r^2}(1,t) + \text{h.o.t.}$ for $\varepsilon \ll 1$,

so obtain $\frac{\partial \phi}{\partial r} = \dot{R} = -i\omega \varepsilon e^{-i\omega t}$ on $r = 1$ after linearizing.

Radiation condition: No incoming waves as $r \to \infty$.

$\phi = e^{-i\omega t} f(r)$ into $\phi_{tt} = \phi_{rr} + \frac{1}{r}\phi_r \Rightarrow -\omega^2 f = f'' + \frac{1}{r} f' \Rightarrow f = AJ_0(\omega r) + BY_0(\omega r)$

Hint $\Rightarrow \phi \sim \left(\frac{2}{\pi \omega r}\right)^{1/2} e^{-i\omega t} \left\{ A\cos\left(\omega r - \frac{\pi}{4}\right) + B\sin\left(\omega r - \frac{\pi}{4}\right)\right\}$

$$\sim \frac{1}{2}\left(\frac{2}{\pi \omega r}\right)^{1/2} e^{-i\omega t} \left\{ \left(A + \frac{B}{i}\right) e^{i(\omega r - \pi/4)} + \left(A - \frac{B}{i}\right) e^{-i(\omega r - \pi/4)} \right\}$$

$$= \frac{1}{(2\pi \omega r)^{1/2}} \left\{ \underbrace{\left(A + \frac{B}{i}\right) e^{i\omega(r-t) - i\pi/4}}_{\text{outgoing wave}} + \underbrace{\left(A - \frac{B}{i}\right) e^{-i\omega(r+t) + i\pi/4}}_{\text{Incoming wave}} \right\} \text{ as } r \to \infty$$

Hence, radiation condition $\Rightarrow A - \frac{B}{i} = 0 \Rightarrow B = iA \Rightarrow \phi = Ae^{-i\omega t}\left(J_0(\omega r) + iY_0(\omega r)\right)$

KBC $\Rightarrow Ae^{-i\omega t}\left(\omega J_0'(\omega r) + i\omega Y_0'(\omega r)\right)\Big|_{r=1} = -i\omega\varepsilon e^{-i\omega t}$

$\Rightarrow A = \frac{-i\varepsilon}{J_0'(\omega r) + iY_0'(\omega r)}$ and hence the required solution.

□

2014

3. A stream flowing over a horizontal surface is governed by the shallow water equations

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0,$$

where $u(x,t)$ is the fluid velocity in the x direction, $h(x,t)$ is the depth of the stream and g is the acceleration due to gravity.

(a) [6 marks] Show that the shallow water equations can be rewritten as

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)(u \pm 2c) = 0,$$

where $c = \sqrt{gh}$, and state the significance of the quantities $Q_{\pm} = u \pm 2c$ and the curves $\mathrm{d}x/\,\mathrm{d}t = u \pm c$.

At $t = 0$, a previously uniform stream $u = 1$, $h = 1$ is suddenly blocked by a dam at $x = 0$, so that $u(x,0) = h(x,0) = 1$, but $u(0,t) = 0$ for $t > 0$. You may assume that $g > 1$.

(b) [12 marks] Concentrate on $x > 0$, $t > 0$. By considering the characteristics that come from $\{t = 0,\ x > 0\}$, and $\{x = 0,\ t > 0\}$, deduce that the value of c immediately downstream of the dam is $c(0,t) = \sqrt{g} - \frac{1}{2}$. Also considering characteristics that come from $\{x = 0,\ t = 0\}$, deduce that the fluid velocity is given by

$$u(x,t) = \begin{cases} 0 & 0 < x < \left(\sqrt{g} - \frac{1}{2}\right)t, \\ \frac{2}{3}\left(\frac{x}{t} - \sqrt{g}\right) + \frac{1}{3} & \left(\sqrt{g} - \frac{1}{2}\right)t < x < \left(\sqrt{g} + 1\right)t, \\ 1 & \left(\sqrt{g} + 1\right)t < x, \end{cases}$$

and find a similar expression for $c(x,t)$.

(c) [4 marks] In $x < 0$, a shock propagates upstream with speed V, separating the uniform stream $u = 1$, $h = 1$ from a region of stationary water with depth $h_0 > 1$ adjacent to the dam. Show that the shock speed is

$$V = \frac{1}{h_0 - 1},$$

and that h_0 satisfies

$$2h_0 = g(h_0 + 1)(h_0 - 1)^2.$$

(d) [3 marks] Combining your solutions from parts (b) and (c), draw a sketch of the height profiles $h(x,t)$ (as a function of x) at $t = 1$ and at $t = 2$, on the same pair of axes.

[*You may assume the conditions for a shallow water shock moving with velocity* $-V$*,*

$$[h(u+V)]_-^+ = \left[h(u+V)^2 + \tfrac{1}{2}gh^2\right]_-^+ = 0,$$

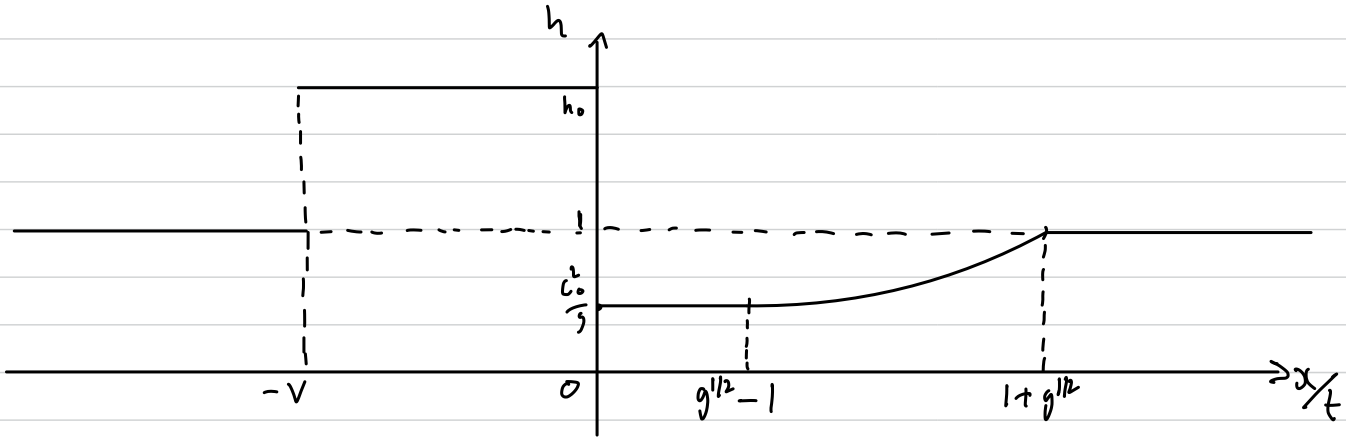
where $[\ ]_-^+$ *denotes the change in the quantity from one side of the shock to the other.*]

2014 Q3(d) Not quite sure how to combine previous parts? Was happy with all previous parts however.

Parts (b) & (c) $\Rightarrow$

$$h_0(x,t) = \begin{cases} 1 & \text{for } x < -Vt \\ h_0 & \text{for } -Vt < x < 0 \\ c_0^2/g & \text{for } 0 < x < (g^{1/2}-\tfrac{1}{2})t \\ \left(\tfrac{x}{t}+2g^{1/2}-1\right)^2/9g & \text{for } (g^{1/2}-\tfrac{1}{2})t < x < (1+g^{1/2})t \\ g^{1/2} & \text{for } x > (1+g^{1/2})t \end{cases}$$

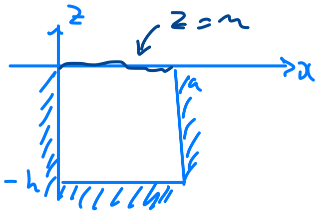
$(c_0 = g^{1/2} - \tfrac{1}{2})$



2017

1. (a) [7 marks] A two-dimensional container has vertical walls at $x = 0$ and $x = a$, a rigid base at $z = -h$, and is filled with fluid of constant density ρ, such that the undisturbed free surface of the fluid would be at $z = 0$. Small-amplitude perturbations disturb this surface to $z = \eta(x, t)$. A gravitational acceleration g acts vertically, and the pressure above the free surface is atmospheric.

Assuming that the flow is irrotational, derive the following linearised model for the velocity potential ϕ and surface displacement η,



$$\nabla^2\phi = 0 \qquad -h < z < 0,$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{and} \quad \frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{on} \quad z = 0,$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h, \qquad \frac{\partial \phi}{\partial x} = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = a.$$

[*You may assume the existence of a velocity potential, and an appropriate version of Bernoulli's equation.*]

(b) [8 marks] Show that separable solutions with temporal frequency $\omega > 0$ must take the form

$$\eta_n(x, t) = \mathrm{Re}\left\{e^{-i\omega t}\cos(k_n x)\right\}, \quad \phi_n(x, z, t) = \mathrm{Re}\left\{-i\omega e^{-i\omega t}\cos(k_n x)\frac{\cosh(k_n(z+h))}{k_n \sinh(k_n h)}\right\},$$

where $k_n = n\pi/a$, for $n = 1, 2, \ldots$. Hence show that the corresponding natural frequencies ω_n are given by

$$\omega_n^2 = g k_n \tanh(k_n h).$$

(c) [10 marks] Suppose now that the container is oscillated from side to side, such that the position of the walls are $x = \epsilon \sin \Omega t$ and $x = a + \epsilon \sin \Omega t$, with $\epsilon \ll a$. Assume that this causes periodic perturbations to the surface displacement that also have frequency Ω.

Write down the new linearized boundary conditions on $x = 0$ and $x = a$.

By writing $\phi(x, z, t) = \mathrm{Re}\left\{U\left(x - \frac{1}{2}a\right)e^{-i\Omega t}\right\} + \tilde{\phi}(x, z, t)$ for a suitable choice of U, and seeking suitable series solutions for η and $\tilde{\phi}$, show that the free surface height is given by

$$\eta(x, t) = \epsilon \sin \Omega t \left[\sum_{m=1}^{\infty} \frac{4\tanh(\tilde{k}_m h)}{\tilde{k}_m a}\frac{\Omega^2}{\Omega^2 - \tilde{\omega}_m^2}\cos(\tilde{k}_m x)\right],$$

where $\tilde{k}_m = k_{2m-1}$ and $\tilde{\omega}_m = \omega_{2m-1}$.

Sketch the free surface at $\Omega t = \pi/2$ when Ω is close to (but not equal to) ω_3.

[*You may make use of the Fourier series,*

$$x - \tfrac{1}{2}a = -\sum_{m=1}^{\infty}\frac{4}{a\tilde{k}_m^2}\cos(\tilde{k}_m x),$$

for $0 < x < a$.]

2017 Q1(c) I'd much appreciate if we could go through the computations.

B5.4/2017/Q1

(a) B – see online notes §2.3 & sheet 1, Q4

(b) Ansatz : $\eta = e^{-i\omega t} X(x)$, $\phi = e^{-i\omega t} X(x) Z(z)$

$$\nabla^2 \phi = 0 \implies \frac{X''}{X} = -\frac{Z''}{Z} = -k^2 \in \mathbb{R}_0^-$$

as trig solutions are required to satisfy $X'(0) = X'(a) = 0$.

$X'' + k^2 X = 0$ with $X'(0) = X'(a) = 0 \implies X = A\cos(k_n x)$, $k_n = \frac{n\pi}{a}$ with $n = 1, 2, \ldots$

$Z'' - k^2 Z = 0$ with $Z'(-h) = 0 \implies Z = B\cosh k(z+h)$

Linearized free surface conditions then give the required dispersion relation.

(c) New BC on $x = 0, a$ is $\phi_x = \mathrm{Re}(\varepsilon \Omega e^{-i\Omega t})$

Show problem for $\tilde{\phi}$ with $U = \varepsilon\Omega$ is the same as for ϕ except for linearized dynamic BC, which becomes

$$\tilde{\phi}_t + g\eta = i\varepsilon\Omega^2 (x - \tfrac{a}{2}) e^{-i\Omega t} = i\varepsilon\Omega^2 e^{-i\Omega t} \sum_{m=1}^{\infty} \frac{4\cos\left((2m-1)\frac{\pi x}{a}\right)}{a(2m-1)^2 \left(\frac{\pi}{a}\right)^2}$$

by the hint. Hence superimpose solns in part (b):

$$\eta = \sum_{n=1}^{\infty} a_n \eta_n, \quad \tilde{\phi} = \sum_{n=1}^{\infty} a_n \phi_n, \text{ but with } \omega = \Omega.$$

This (formally) satisfies everything except the inhomogeneous dynamic BC above.

Substituting for η & $\tilde{\phi}$ & equating Fourier coeffs gives the a_n's and hence the series solution. Note that the $m = 2$ mode dominates for Ω close to ω...

(c) KBC $\Rightarrow$ $\phi_x = \varepsilon \Omega e^{-i\Omega t}$ on $x = 0, a$ (RPU) after linearization.

$$\phi = \underset{u}{\underline{\varepsilon\Omega}}(x - \tfrac{a}{2})e^{-i\Omega t} + \tilde{\phi}, \text{ then } \nabla^2\tilde{\phi} = 0 \text{ in } 0<x<a, -h<z<0,$$

with $\tilde{\phi}_x = 0$ on $x = 0, a$, $\tilde{\phi}_z = 0$ on $z = -h$

and $\tilde{\phi}_z = \eta_t$, $\tilde{\phi}_t + g\eta \overset{(*)}{=} i\varepsilon\Omega^2(x - \tfrac{a}{2})e^{-i\Omega t}$ on $z = 0$

i.e. same as homogeneous problem except for inhomogeneous BC

Hence, superimpose sep. sol$^{\text{ns}}$ with $\omega = \Omega$ $\Rightarrow$

$$\eta = \sum_{n=1}^{\infty} a_n e^{-i\Omega t}\cos k_n x, \quad \tilde{\phi} = \sum_{n=1}^{\infty} -a_n i\Omega e^{-i\Omega t}\cos k_n x\,\frac{\cosh k_n(z+h)}{k_n \sinh k_n h} \quad \text{(RPU)}$$

where $k_n = \pi n/a$.

This satisfies all but $(*)$ by construction.

$$(*) \Rightarrow \sum_{n=1}^{\infty}\left\{-\Omega^2\frac{1}{k_n\tanh k_n h} + g\right\}a_n\cos k_n x = i\varepsilon\Omega^2(x - \tfrac{a}{2})$$

$$\Rightarrow \sum_{n=1}^{\infty}\left\{1 - \frac{\Omega^2}{\omega_n^2}\right\}g a_n \cos\frac{n\pi x}{a} = i\varepsilon\Omega^2\sum_{n=1}^{\infty}\frac{2a}{n^2\pi^2}((-1)^n - 1)\cos\frac{n\pi x}{a} \quad \text{by hint}$$

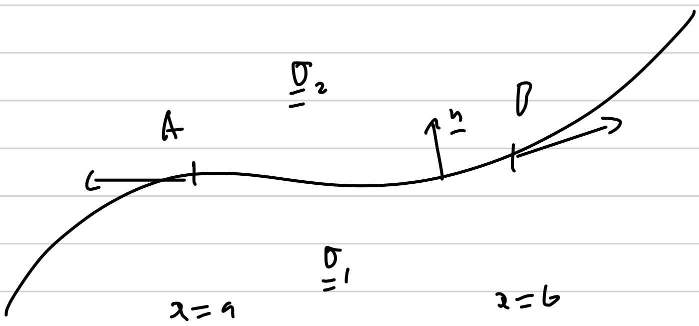
where $\omega_n^2 = gk_n\tanh k_n h$

Equate coefficients $\Rightarrow$ $\dfrac{(\omega_n^2 - \Omega^2)}{\omega_n^2}g a_n = i\varepsilon\Omega^2\dfrac{2a}{n^2\pi^2}((-1)^n - 1)$ for $n \in \mathbb{N}^+$

Substitute into series and take real part $\Rightarrow$ given solution for η. □

NB: Assuming $\Omega^2 \neq \omega_n^2$ $\forall_n$!!!

Force balance on a free surface with surface tension γ



$$\int_A^B \underline{\underline{\sigma}}_1 \cdot \underline{n} + \underline{\underline{\sigma}}_2 \cdot (-\underline{n})\, ds$$

$$+ \gamma \underline{t}\big|_B - \gamma \underline{t}\big|_A = \underline{0}$$

$$\Rightarrow \int_A^B \left[\underline{\underline{\sigma}} \cdot \underline{n}\right]_1^2 - \frac{d}{ds}(\gamma \underline{t})\, ds = \underline{0}$$

$$\Rightarrow \left[\underline{\underline{\sigma}} \cdot \underline{n}\right]_1^2 = \frac{d}{ds}(\gamma \underline{t})$$

BS.4 : $\underline{\underline{\sigma}}_i = -p_i \underline{\underline{I}}$

2015

2. A barotropic gas with pressure-density relation $p = P(\rho)$ flows steadily past a thin symmetric wing, which occupies the region $|x| \leqslant a$ and has upper and lower surfaces given by $y = \pm f(x)$, where $f(\pm a) = 0$ and $|f| \ll a$. The flow is governed by Euler's equations for an inviscid compressible fluid,

$$\nabla \cdot (\rho \mathbf{u}) = 0, \qquad \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p.$$

Far upstream of the wing the flow is uniform, with speed U in the x direction, density ρ_0, pressure $p_0 = P(\rho_0)$, and sound speed given by $c_0^2 = P'(\rho_0)$.

(a) [10 marks] Assuming the flow is irrotational and writing $\mathbf{u} = U\hat{\mathbf{e}}_x + \nabla\phi$, find a linearised expression for the pressure perturbation in terms of the velocity potential, and show that $\phi(x, y)$ satisfies

$$(M^2 - 1)\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2},$$

where the Mach number M should be defined. Also derive the linearised boundary conditions

$$\frac{\partial \phi}{\partial y} = \pm U f'(x) \quad \text{on} \quad y = 0\pm, \ |x| \leqslant a,$$

where $y = 0\pm$ denotes the boundary approached from above and below, respectively.

Briefly describe the conditions that should be applied at infinity if $M < 1$, and explain why an appropriate condition in the case $M > 1$ is $\phi \to 0$ as $x \to -\infty$.

(b) [8 marks] Suppose now that $M > 1$. With the aid of a diagram to identify distinct regions of the flow, find the solution for $\phi(x, y)$.

(c) [7 marks] Suppose further that $f(x) = b(1 - x^2/a^2)$. Sketch the streamlines of the flow, and calculate the drag force D on the wing. Show that D is minimised (for $M > 1$) when $U = \sqrt{2}\, c_0$.

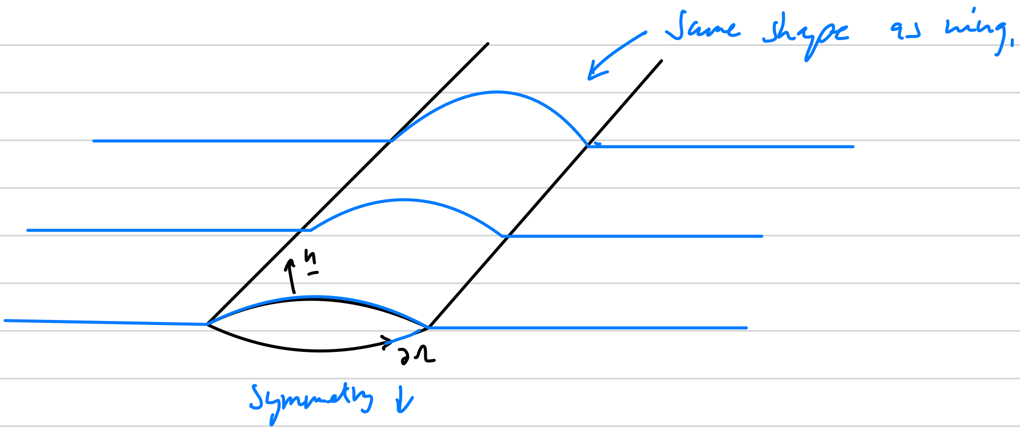
[*You may assume the expression*

$$D = \int_{-a}^{a} f'(x)\,[p(x, 0-) + p(x, 0+)]\,\mathrm{d}x,$$

for the drag force on the wing.]

Would it be possible to go over 2015 q2c please?

2015/Q2(c)



Force on wing $\underline{F} = \int_{\partial\Omega} -p\,\underline{n}\,ds$, $\underline{n} = \dfrac{(\mp f'(x),\ 1)}{\sqrt{1+f'(x)^2}}$

But given $D = \int_{-a}^{a} f'(x)\big(p(x,0-) + p(x,0+)\big)\,dx$ after linearization.

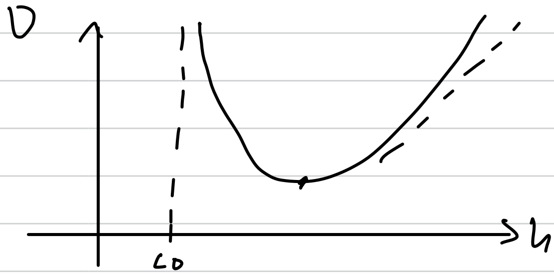
$$= -\rho_0 U \int_{-a}^{a} f'(x)\big(\phi_x(x,0-) + \phi_x(x,0+)\big)\,dx \qquad (p = -\rho_0 U \phi_x)$$

$$= +\frac{\rho_0 U^2}{\beta}\int_{-a}^{a} 2 f'(x)^2\,dx \qquad (\text{by part (b)}) \quad (\beta = \sqrt{M^2-1})$$

$$= \frac{\rho_0 U^2}{\beta}\int_{-a}^{a} 2b^2\left(-\frac{2x}{a^2}\right)^2 dx$$

$$= \frac{8\rho_0 U^2}{\beta}\,\frac{b^2}{a^4}\,\frac{2a^3}{3}$$

i.e. $$D = \frac{16}{3}\,\frac{\rho_0 b^2 U^2}{a\sqrt{\frac{U^2}{c_0^2}-1}}$$



Want min of D for $U > c_0$.

$$\frac{dD}{dU^2} = \frac{16}{3}\,\frac{\rho_0 b^2}{a}\left[\frac{1}{\sqrt{\frac{U^2}{c_0^2}-1}} - \frac{U^2}{2\left(\frac{U^2}{c_0^2}-1\right)^{3/2} c_0^2}\right] = 0$$

$$\Rightarrow \quad 2\left(\frac{U^2}{c_0^2}-1\right) = \frac{U^2}{c_0^2} \quad \Rightarrow \quad U^2 = 2c_0^2 \quad \text{or} \quad U = \sqrt{2}\,c_0. \qquad \square$$