Consultation session questions

- A quick question about supersonic flow past a thin wing. On page 60 of the notes, where we're finding the value of ϕ in each of the 6 regions, in regions 3 and 6, is ϕ supposed to be 0 or a constant (so that we have continuity between regions 2 and 3, 5 and 6)? The notes say that ϕ should be 0 (so that they are zones of silence) but in the lectures, you imposed continuity so that ϕ are non-zero constants instead, so I was just wondering which one I should use? Can we still say that there are zones of silence if ϕ is non-zero?
- 2012 Q1(b)(iv) Could you talk through whether the waves are dispersive?
- 2012 Q2(b) Could you just talk through how to apply FT for this question?
- 2021 Q3(c)(ii) Unsure of how to approach this part!!
- 2013 Q1(c) Could you talk through the BCs we need to apply here?
- 2013 Q2(a)(iii) Could you just quickly cover region of silence?
- ✓ 2013 Q2(b) I think I got the correct BCs and solution, but could I perhaps check these?
- 2013 Q2(c)(ii) Less sure on this part. Behaviour changes at y=0 but not sure what this means!
- 2016 Q3(b) Why must the fluid depths and velocities satisfy these relations to the left and the right of x = Vt? How come the shock / expansion fan doesn't at x = bt doesn't get in the way? Why do we know that the shock is a straight line?
- 2017 Q2(c,d) Please could you go over them.
- 2017 Q3(c) Please could you go over it.
- 2014 Q1(c) We will end up with $AJ_0 + BY_0$ as our solution, but could you just discuss the BCs we need here?
- 2014 Q3(d) Not quite sure how to combine previous parts? Was happy with all previous parts however.
- 2017 Q1(c) I'd much appreciate if we could go through the computations.
- ✓ Is contour integration examinable? Yes but very unlikely!
- ✓ 2014 Q3(d) Please can we go through the sketches.
- ✓ 2015 Q2(c), Q3(b) and Q3(c) Please can we go through the sketches.

2020

1. The flow of a stratified incompressible fluid with density ρ, velocity $\mathbf{u}$ and pressure p is governed by

$$\frac{\partial \rho}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\rho = 0, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0, \quad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right) = -\boldsymbol{\nabla} p - \rho g \mathbf{k},$$

where g the acceleration due to gravity.

(a) [7 marks] (i) Determine the pressure $p = p_0(z)$ in the stationary state in which the fluid is at rest with density $\rho_0(z) = \rho_a e^{-\beta z}$ and $p = p_a$ on $z = 0$, where $\rho_a > 0$, β and p_a are constants.

(ii) Small amplitude waves perturb the stationary state so that $\rho = \rho_0(z) + \rho'$, $p = p_0(z) + p'$ and $\mathbf{u} = u'\mathbf{i} + v'\mathbf{j} + w'\mathbf{k}$, where the primed variables are small. Write down the linearized versions of the governing equations and deduce that w' satisfies

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right) = -\beta g\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} - \frac{1}{g}\frac{\partial^3 w'}{\partial z \partial t^2}\right). \qquad (\star)$$

(b) [6 marks] Suppose that the flow is two dimensional and gravity dominated with the fluid occupying the channel $x > 0$, $0 < z < h$ between rigid walls at $z = 0$ and $z = h$, so that $w'(x, z, t)$ is governed by

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) = -\beta g \frac{\partial^2 w'}{\partial x^2} \quad \text{for} \quad x > 0,\ 0 < z < h,$$

with $w' = 0$ on $z = 0$, h for $x > 0$. A wave-maker at $x = 0$ disturbs the fluid such that

$$w' = a e^{-i\Omega t}\sin\left(\frac{\pi z}{h}\right) \quad \text{on} \quad x = 0,\ 0 < z < h,$$

where a and $\Omega > 0$ are constants. By imposing as $x \to \infty$ either boundedness of w' or a radiation condition on w', as appropriate, solve for w' when (i) $\Omega^2 > \beta g$ and (ii) $\Omega^2 < \beta g$.

(c) [12 marks] Now suppose that the flow is three dimensional with $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ below a free surface at $z = \eta(x, y, t)$ on which

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}, \quad p - p_a = -\gamma\kappa,$$

where γ is the surface tension and κ is the curvature. The primed variables and η are small, so that $w'(x, y, z, t)$ is governed by $(\star)$ for $z < 0$.

(i) Assuming that $\kappa = \boldsymbol{\nabla}^2\eta$ after linearization for small η, derive the linearized boundary condition

$$\rho_a \frac{\partial^3 w'}{\partial z \partial t^2} = \rho_a g\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2}\right) - \gamma\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w' \quad \text{on} \quad z = 0.$$

(ii) Show that there are waves of the form

$$w' = A\exp\big(i(kx\cos\alpha + ky\sin\alpha - \omega t) + \lambda z\big)$$

with A, k, α, ω and λ being constant and $\mathrm{Re}(\lambda) \geqslant 0$ only if the frequency ω is such that ω^2 is a function of the wavenumber k that you should determine.

2020/B5.4/Q1

(a)(i) All equations satisfied with $\underline{u} = \underline{0}$, $\rho_0 = \rho_a e^{-\beta z}$, $p = p_0(z)$ provided $\nabla p_0 = -\rho_0 g \underline{k}$ in $z < 0$ with $p_0 = p_a$ on $z = 0$.

[B2] Hence, $p_0(z) = p_a - \int_0^z \rho_0(s) g\, ds = p_a + \frac{\rho_a g}{\beta}\left(e^{-\beta z} - 1\right)$.

(a)(ii) The linearized PDEs are (with $\underline{u}' = u'\underline{i} + v'\underline{j} + w'\underline{k}$)

$$\rho'_t + w'\frac{d\rho_0}{dz} \underset{(1)}{=} 0, \quad \nabla\cdot\underline{u}' \underset{(2)}{=} 0, \quad \rho_0 \underline{u}'_t \underset{(3)}{=} -\nabla p' - \rho' g \underline{k}$$

Eliminate u' & v':

$$p'_{xx} + p'_{yy} = -\rho_0 u'_{tx} - \rho_0 v'_{ty} \qquad \text{(by (3))}$$
$$= \rho_0 w'_{zt} \qquad \text{(by (2))}$$

Since $\frac{d\rho_0}{dz} = -\beta\rho_0$, we are left with

$$\rho'_t \underset{(4)}{=} \beta\rho_0 w', \quad p'_{xx} + p'_{yy} \underset{(5)}{=} \rho_0 w'_{zt}, \quad \rho_0 w'_t \underset{(6)}{=} -p'_z - \rho' g$$

Now eliminate ρ' and p':

$$(\rho_0 w'_{zt})_{zt} = (\partial_x^2 + \partial_y^2)\, p'_{zt} \qquad \text{(by (5))}$$
$$= (\partial_x^2 + \partial_y^2)(-\rho_0 w'_{tt} - \rho'_t g) \qquad \text{(by (6))}$$

$$\Rightarrow \rho_0 w'_{zztt} + \frac{d\rho_0}{dz} w'_{ztt} = (\partial_x^2 + \partial_y^2)(-\rho_0 w'_{tt} - \beta g \rho_0 w') \qquad \text{(by (4))}$$

[B5]

$$\Rightarrow (w'_{xx} + w'_{yy} + w'_{zz})_{tt} \underset{(*)}{=} -\beta g \left(w'_{xx} + w'_{yy} - \frac{1}{g} w'_{ztt}\right)$$

(as $\frac{d\rho_0}{dz} = -\beta\rho_0$)

(b) $w' = e^{-i\lambda t} f(x) \sin\left(\frac{\pi z}{h}\right) \Rightarrow (-i\lambda)^2\left(f'' - \frac{\pi^2}{h^2} f\right) = -\beta g f''$

or $f'' + \dfrac{\pi^2}{h^2\left(\frac{\beta g}{\lambda^2} - 1\right)} f = 0$ for $x > 0$ and $\lambda^2 \neq \beta g$.

(i) $\Omega^2 > \beta g \Rightarrow f'' - \nu^2 f = 0, \quad \nu = \dfrac{\pi}{h\left(1-\frac{\beta g}{\Omega^2}\right)^{1/2}} > 0$

$$\Rightarrow f = Ae^{\nu x} + Be^{-\nu x} \quad (A, B \in \mathbb{C} \text{ arb.})$$

BC on $x=0$ & w' bdd at ∞ $\Rightarrow f(0)=a,\ |f(\infty)|<\infty \Rightarrow A=0,\ B=a$

Hence, $w' = a e^{-i\Omega t - \nu x}\sin\left(\frac{\pi z}{h}\right)$, ν as above.

(ii) $\Omega^2 < \beta g \Rightarrow f'' + \mu^2 f = 0, \quad \mu = \dfrac{\pi}{h\left(\frac{\beta g}{\Omega^2}-1\right)^{1/2}} > 0$

$$\Rightarrow f = Ae^{i\mu x} + Be^{-i\mu x} \quad (A, B \in \mathbb{C} \text{ arb.})$$

Hence, $w' = \left(A e^{i(\mu x-\Omega t)} + B e^{-i(\mu x+\Omega t)}\right)\sin\left(\frac{\pi z}{h}\right)$

BC on $x=0$ & radiation condition (no inward waves) $\Rightarrow f(0)=a,\ B=0$

[S6] Hence $w' = a e^{i(\mu x-\Omega t)}\sin\left(\frac{\pi z}{h}\right)$, μ as above

(c)(i) BCs $\Rightarrow w' = \eta_t + u'\eta_x + v'\eta_y,\ p_0(\eta) - p_a + p' = -\gamma\kappa$ on $z=\eta$.

For small η, Taylor's Theorem gives

$$w'|_{z=\eta} = w'|_{z=0} + \eta w'_z|_{z=0} + \text{h.o.t.}$$

$$p_0(\eta) = p_0(0) + \eta \frac{dp_0}{dz}(0) + \text{h.o.t.}$$

and similarly for u', v' and p'. Hence, the linearized BCs are

$$w' \underset{(7)}{=} \eta_t, \quad p' \underset{(8)}{=} \rho_a g\eta - \gamma(\eta_{xx}+\eta_{yy}) \text{ on } z=0$$

where we used the fact $\kappa = \nabla^2\eta$ after linearizing for small η and $p_0(0) = p_a$, $\frac{dp_0}{dz}(0) = -\rho_0(0)g = -\rho_a g$.

Eliminate η & p' :

$$\rho_a w'_{ztt} = (\partial_x^2 + \partial_y^2) p'_t \qquad (\text{by } ⑤ \text{ on } z=0)$$

$$= \rho_a g (\partial_x^2 + \partial_y^2)(\eta_t - \Gamma(\partial_x^2 + \partial_y^2)\eta_t) \qquad (\text{by } ⑧, \Gamma = \tfrac{\gamma}{\rho_a g})$$

$$= \rho_a g (\partial_x^2 + \partial_y^2)(w' - \Gamma(\partial_x^2 + \partial_y^2) w') \qquad (\text{by } ⑦)$$

[S|NG] giving $\rho_a w'_{ztt} \underset{(**)}{=} \rho_a g (\partial_x^2 + \partial_y^2) w' - \gamma (\partial_x^2 + \partial_y^2)^2 w'$ on $z = 0$

(c)(ii) Substituting $w' = A \exp\{ik\cos\alpha\, x + ik\sin\alpha\, y - i\omega t + \lambda z\}$,

$$(*) \Rightarrow (-i\omega)^2\left((ik\cos\alpha)^2 + (ik\sin\alpha)^2 + \lambda^2\right) = -\beta g\left((ik\cos\alpha)^2 + (ik\sin\alpha)^2 - \frac{\lambda(-i\omega)^2}{g}\right)$$

$$(**) \Rightarrow \frac{1}{g}\lambda(-i\omega)^2 = (ik\cos\alpha)^2 + (ik\sin\alpha)^2 - \Gamma\left((ik\cos\alpha)^2 + (ik\sin\alpha)^2\right)^2$$

Hence, $\omega^2(k^2 - \lambda^2 + \beta\lambda) = \beta g k^2$, $\lambda\omega^2 \underset{(+)}{=} g k^2(1 + \Gamma k^2)$

Eliminate ω^2 : $(1 + \Gamma k^2)(k^2 - \lambda^2 + \beta\lambda) = \beta\lambda$

$$\Rightarrow \lambda^2 - \frac{\beta\Gamma k^2}{1 + \Gamma k^2}\lambda - k^2 = 0, \quad 2\lambda \underset{(++)}{=} \frac{\beta\Gamma k^2 \pm \left((\beta\Gamma k^2)^2 + 4(1+\Gamma k^2)^2 k^2\right)^{1/2}}{1 + \Gamma k^2}$$

But only the +ve root is admissible for $Re(\lambda) \geq 0$, so that (+) and (++) give the dispersion relation

$$\left\{\beta\Gamma|k| + \left(4(1+\Gamma k^2)^2 + \beta^2\Gamma^2 k^2\right)^{\frac{1}{2}}\right\}\omega^2 = 2g|k|(1 + \Gamma k^2)^2,$$

[NG] where we divided through by $|k| = \sqrt{k^2}$.

Thus, $$\omega^2 = \frac{2g|k|(1+\Gamma k^2)^2}{\beta\Gamma|k| + \left(4(1+\Gamma k^2)^2 + \beta^2\Gamma^2 k^2\right)^{1/2}}$$

[N|SG] where $\Gamma = \gamma / \rho_a g$.

2020

2. (a) [10 marks] Small-amplitude waves disturb a barotropic gas contained in a two-dimensional rectangular pipe between rigid walls at $x = 0$, $x = a$ and free ends at $y = 0$, $y = b$ maintained at constant pressure p_0. You may assume that the linearized equations for the potential $\phi(x, y, t)$ for the velocity pertubations and for the pressure $p(x, y, t)$ are given by

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right), \qquad p = p_0 - \rho_0 \frac{\partial \phi}{\partial t}$$

in $0 < x < a$, $0 < y < b$, where the sound speed c_0 and undisturbed density ρ_0 are constant.

(i) State the boundary conditions for ϕ on the boundary of the pipe.

(ii) By seeking a separable solution of the form $\phi = \mathrm{e}^{-\mathrm{i}\omega t} F(x) G(y)$, find the normal modes and the natural frequencies ω.

(b) [15 marks] Small deflections $\eta(x, t)$ of an elastic beam lying on a shallow water layer satisfy the equation

$$\frac{\partial^2 \eta}{\partial t^2} = \beta^2 \frac{\partial^6 \eta}{\partial x^6} \quad \text{for} \quad -\infty < x < \infty,\ t > 0,$$

where β is a positive constant. The beam is released from rest such that

$$\eta(x, 0) = \frac{\alpha}{\epsilon \sqrt{\pi}} \mathrm{e}^{-x^2/\epsilon^2}, \qquad \frac{\partial \eta}{\partial t}(x, 0) = 0 \quad \text{for} \quad -\infty < x < \infty,$$

where α and ϵ are positive constants.

(i) Show that the solution may be written in the form

$$\eta(x, t) = \int_{-\infty}^{\infty} F(k) \cos\left(\omega(k) t\right) \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k,$$

where the function $F(k)$ should be defined and $\omega(k) = \beta |k|^3$.

[*You may assume that the Fourier transform of* $\mathrm{e}^{-x^2/\epsilon^2}$ *is* $\epsilon\sqrt{\pi}\mathrm{e}^{-\epsilon^2 k^2/4}$.]

(ii) Use the method of stationary phase to show that if ϵ is sufficiently small then

$$\eta(x, t) \sim \frac{A}{(xt)^{1/4}} \cos\left(\frac{B x^{3/2}}{t^{1/2}} - \frac{\pi}{4} \right)$$

as $x, t \to +\infty$ with x/t held constant, where the constants A and B should be determined.

[*You may assume that if $f(k)$ and $\psi(k)$ are sufficiently well behaved, and $\psi(k)$ is real-valued and such that $\psi'(k)$ has a single zero at k_*, then*

$$\int_{-\infty}^{\infty} f(k) \mathrm{e}^{\mathrm{i}\psi(k) t}\, \mathrm{d}k \sim f(k_*) \mathrm{e}^{\mathrm{i}(\psi(k_*) t \pm \pi/4)} \sqrt{\frac{2\pi}{|\psi''(k_*)|\, t}} \quad \text{as} \quad t \to \infty,$$

where the $\pm$ takes the sign of $\psi''(k_*)$.]

2020/B5.4/Q2

+1 (a)(i) No normal flow through rigid walls $\Rightarrow \phi_x = 0$ on $x = 0, a$.

Since $p = p_0 - \rho_0 \phi_t$, the condition of constant pressure p_0 on the free ends $\Rightarrow \phi_t = 0$ on $y = 0, b$.

+2 [B2]

(a)(ii) Seek nontrivial separable solution $\phi = e^{-i\omega t} F(x) G(y)$.

Wave equation $\Rightarrow$ $-\frac{\omega^2}{c_0^2} - \frac{F''(x)}{F(x)} = \frac{G''(y)}{G(y)}$ $(FG \neq 0)$

+1

LHS ind. y & RHS ind. x $\Rightarrow$ LHS = RHS ind. x & y and hence constant.

+1 +2

Hence, $\frac{F''}{F} = -\lambda^2$, $\frac{G''}{G} = -\mu^2$, $\omega^2 = c_0^2(\lambda^2 + \mu^2)$,

where $\lambda, \mu \in \mathbb{R}$ and the signs of the constants have been chosen to give the oscillatory solutions required for nontrivial solutions for which the BCs in (a)(i) require $F'(0) = F'(a) = 0$, $G(0) = G(b) = 0$.

+1 +2

+1 [B/S4]

For $\lambda = 0$, F = constant, but for $\lambda \neq 0$, $F = A\cos\lambda x + B\sin\lambda x$ ($A, B \in \mathbb{C}$ arb.), so BCs $\Rightarrow B = 0$ and $\lambda a = m\pi$ with $m \in \mathbb{Z} \setminus \{0\}$. Combo $\Rightarrow F \propto \cos\left(\frac{m\pi x}{a}\right)$, $\lambda = \frac{m\pi}{a}$, $m \in \mathbb{Z}$.

+1 +2

For $\mu = 0$, $G = 0$, but for $\mu \neq 0$, $G = C\cos\mu y + D\sin\mu y$ ($C, D \in \mathbb{R}$ arb.), so BCs $\Rightarrow C = 0$, $\mu b = n\pi$ with $n \in \mathbb{Z} \setminus \{0\}$. Hence, $G \propto \sin\left(\frac{n\pi y}{b}\right)$, $\mu = \frac{n\pi}{b}$, $n \in \mathbb{Z}\setminus\{0\}$.

+1 +1

Combo $\Rightarrow$ normal modes are $\phi = E e^{-i\omega t} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$ ($E \in \mathbb{C}$ arb.) with natural frequencies ω s.t.

+1 +1

$$\omega^2 = c_0^2 \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \quad \text{for } m \in \mathbb{Z},\ n \in \mathbb{Z}\setminus\{0\}.$$

[B/S4]

(b)(i) Fourier transform in x : $\hat{\eta}(k,t) = \int_{-\infty}^{\infty} \eta(x,t) e^{-ikx} dx$.

PDE $\Rightarrow$ $\hat{\eta}_{tt} = \beta^2 (ik)^6 \hat{\eta} = -\beta^2 k^6 \hat{\eta}$ for $t > 0$.

+3 ICs $\underset{\text{(hint)}}{\Rightarrow}$ $\hat{\eta} = \alpha e^{-\varepsilon^2 k^2/4}$, $\hat{\eta}_t = 0$ at $t = 0$.

Hence, $\hat{\eta}(k,t) = A(k)\cos(\omega(k)t) + B(k)\sin(\omega(k)t)$, where $A(k)$, $B(k)$ are arbitrary and $\omega^2 = \beta^2 k^6$, so that $\omega(k) = \beta |k|^3$ wlog.

+2 ICs $\Rightarrow$ $A(k) = \alpha e^{-\varepsilon^2 k^2/4}$, $B(k) = 0$.

Inverse Fourier transform $\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\eta}(k,t) e^{ikx} dk$.

Hence, $\eta(x,t) = \int_{-\infty}^{\infty} F(k)\cos(\omega(k)t) e^{ikx} dk$,

+1

[S6] where $F(k) = \frac{\alpha}{2\pi} e^{-\varepsilon^2 k^2/4}$ and $\omega(k) = \beta |k|^3$.

(b)(ii) Let $\frac{x}{t} = V = O(1)$ as $x, t \to +\infty$ and use $\cos(\omega t) = \frac{e^{-i\omega t} + e^{i\omega t}}{2}$

$\Rightarrow$ $\eta(Vt,t) = \int_{-\infty}^{\infty} \frac{1}{2} F(k) \left(e^{-i\omega t} + e^{+i\omega t} \right) e^{ikVt} dk$

$\Rightarrow$ $\eta(Vt,t) = \frac{1}{2}\left(I_+(t) + I_-(t) \right)$, where

$$I_\pm(t) = \int_{-\infty}^{\infty} F(k) e^{i\psi_\pm(k)t} dk,$$

+2 $$\psi_\pm(t) = kV \mp \omega(k).$$

[B2] Can now apply the method of stationary phase to $I_\pm(t)$.

The main contribution to $I_\pm(t)$ as $t \to +\infty$ comes from wavenumbers $k = k_*^\pm$, where the phase $\psi_\pm(k)$ is stationary, i.e. $\psi_\pm'(k_*^\pm) = 0$.

Calculate: $\omega = \beta|k|^3 \Rightarrow \omega'(k) = 3\beta k|k|,\ \omega''(k) = 6\beta|k|$.

Hence $\psi_\pm'(k_*^\pm) = 0 \Rightarrow V \mp \omega'(k_*^\pm) = 0 \Rightarrow 3\beta k_*^\pm|k_*^\pm| = \pm V$, giving the critical wavenumbers $k_*^\pm = \pm\left(\frac{V}{3\beta}\right)^{1/2}$

These are each simple zeros because $\psi_\pm''(k_*^\pm) = \mp 6\beta|k_*^\pm| \neq 0$ and $\psi_\pm$ is twice ctsly diff, while $F(k)$ is ∞ diff and decays exponentially rapidly as $k \to \pm\infty$, so hint applies giving

$$I_\pm(t) \sim F(k_*^\pm)\exp\left\{i\left(\psi_\pm(k_*^\pm)t \mp \frac{\pi}{4}\right)\right\}\left(\frac{2\pi}{|\psi_\pm''(k_*^\pm)|t}\right)^{1/2}$$

as $t \to \infty$ because $\operatorname{sgn}(\psi_\pm''(k_*^\pm)) = \mp 1$

Calculate: $F(k_*^\pm) = \frac{\alpha}{2\pi}e^{-\varepsilon^2V^2/12\beta}$

$$\psi_\pm(k_*^\pm) = \pm\frac{V^{3/2}}{(3\beta)^{1/2}} \mp \beta\left|\frac{V^{1/2}}{(3\beta)^{1/2}}\right|^3 = \pm\frac{2V^{3/2}}{3(3\beta)^{1/2}}$$

$$|\psi_\pm''(k_*^\pm)| = 6\beta\left(\frac{V}{3\beta}\right)^{1/2} = 2(3\beta V)^{1/2}$$

Hence, $I_\pm(t) \sim \frac{\alpha}{2\pi}e^{-\varepsilon^2V^2/12\beta}\exp\left\{\pm i\left(\frac{2V^{3/2}t}{3(3\beta)^{1/2}} - \frac{\pi}{4}\right)\right\}\left(\frac{2\pi}{2(3\beta V)^{1/2}t}\right)^{\frac{1}{2}}$

as $t \to \infty$, giving

$$n(Vt,t) \sim \frac{\alpha e^{-\varepsilon^2V^2/12\beta}}{(4\pi(3\beta V)^{1/2}t)^{1/2}}\cos\left(\frac{2V^{3/2}t}{3(3\beta)^{1/2}} - \frac{\pi}{4}\right)$$

Finally, $e^{-\varepsilon^2V^2/12\beta} \sim 1$ for ε sufficiently small and replace V with x/t elsewhere to obtain

$$n(x,t) \sim \frac{A}{(xt)^{1/4}}\cos\left(\frac{2x^{3/2}}{3(3\beta t)^{1/2}} - \frac{\pi}{4}\right)$$

[N|S7] as $x,t \to +\infty$ with $\frac{x}{t} = O(1)$, where $A = \frac{\alpha}{(4\pi)^{1/2}(3\beta)^{1/4}}$.

3. (a) [12 marks] Shallow water of uniform depth h_0 is held at rest in $x < 0$ by a dam at $x = 0$. The dam is suddenly removed at time $t = 0$, so that the water flows into $x > 0$ forming a shallow layer of depth $h(x,t)$ and horizontal fluid velocity $u(x,t)$ under the action of gravity with acceleration g. You may assume that the *Riemann invariants* $u \pm 2c$ are conserved along *characteristics* satisfying $\mathrm{d}x/\,\mathrm{d}t = u \pm c$, where the wave speed $c = \sqrt{gh}$.

(i) Show that

$$u = 0, \quad c = c_0 \quad \text{for} \quad x < -c_0 t,\ t > 0,$$

where $c_0 = \sqrt{gh_0}$. Derive the solution given by

$$u = \frac{2}{3}\left(c_0 + \frac{x}{t}\right), \quad c = \frac{1}{3}\left(2c_0 - \frac{x}{t}\right) \quad \text{for} \quad -c_0 t < x < 2c_0 t,\ t > 0,$$

explaining why it is not valid for $x > 2c_0 t$, $t > 0$.

(ii) Find the time taken by a fluid element initially at $x = -a < 0$ to reach $x = 0$.

(b) [13 marks] Now suppose that shallow water of uniform depth h_0 is held at rest in $x > 0$ by a dam at $x = 0$. At time $t = 0$ the dam suddenly starts to move into the water and to leak, so that $h(u - U) = -\lambda h^2$ on the dam at $x = Ut$ for $t > 0$, where U and λ are positive constants. Suppose that the flow separates into a region of uniform velocity and uniform depth βh_0 in $Ut < x < Vt$ and a stationary region of uniform depth h_0 in $x > Vt$, with a shock at $x = Vt$ moving at constant speed V. You may assume the *Rankine-Hugoniot conditions*,

$$[h(u - V)]_-^+ = 0, \qquad \left[h(u - V)^2 + \frac{gh^2}{2}\right]_-^+ = 0,$$

where $[\ ]_-^+$ denotes the change in the quantity from one side of the shock to the other.

(i) Find the shock speed V in terms of β and $c_0 = \sqrt{gh_0}$, and deduce that

$$\frac{(\beta - 1)\sqrt{\beta + 1}}{\sqrt{2\beta}} = \frac{U}{c_0} - \frac{\lambda h_0}{c_0}\beta.$$

Using a diagram explain why this equation has a unique positive root for β.

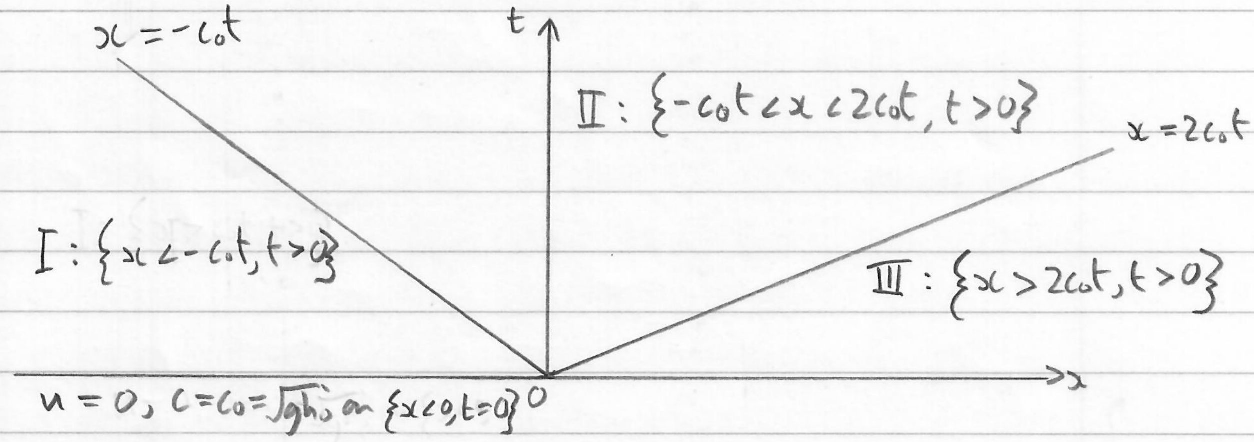
(ii) Show from first principles that the net rate at which energy flows out of the moving shock is given by

$$\rho g h_0^2 V q(\beta),$$

where ρ is the density and $q(\beta)$ is a function of β that you should determine. Hence explain why the condition $U > \lambda h_0$ is necessary for the shock to be physical.

B5.4/2020/Q3

(a)(i)



Region I: Where $C_\pm$ char^{cs} from $\{x<0,\ t=0\}$ intersect, $u \pm 2c = \pm 2c_0 \Rightarrow u=0,\ c=c_0$. Hence such $C_\pm$ char^{cs} are straight lines with $dx/dt = \pm c_0$ and therefore map

[B2] out $x<-c_0t$, $t>0$, i.e. $u=0,\ c=c_0$ in region I.

Region II: Where a C_- char^{c} intersects the family of C_+ char^{cs} from $\{x<0,\ t=0\}$, we have $u-2c=$ const. and $u+2c=2c_0$ on the C_- char^{c}; hence u and c are constant on it and it is therefore straight. To avoid it crossing other C_- char^{cs} in regions I and II, it must originate from the origin and therefore have slope $x/t = u-c$. Solving $u+2c=2c_0$, $u-c=x/t$ gives $u=\frac{2}{3}(c_0+x/t)$ and $c=\frac{1}{3}(2c_0-x/t)$ in region II, which therefore maps out $-c_0t<x<2c_0t$, $t>0$ because we require $c,h \geq 0$, i.e. expansion fan terminates on $x=2c_0t$ where $c=h=0$.

[B|S5] Region III: u and c undefined as there is no water.

(a)(ii) The fluid element at $x=-a<0$ at $t=0$ moves with the flow according to the ODE

$$\frac{dx}{dt} = u(x,t) = \begin{cases} 0 & \text{for } x<-c_0t,\ t>0, \\ \frac{2}{3}\left(c_0+\frac{x}{t}\right) & \text{for } -c_0t<x<2c_0t,\ t>0. \end{cases}$$

Hence the element does not move with $x = -a$ until $t = a/c_0$. After this it moves according to

$$\frac{dx}{dt} - \frac{2}{3t}x = \frac{2}{3}c_0 \Rightarrow \frac{d}{dt}\left(\frac{x}{t^{2/3}}\right) = \frac{2c_0}{3t^{2/3}}$$

with $x = -a$ at $t = a/c_0$; thus $\frac{x}{t^{2/3}} = 2c_0 t^{1/3} + A$

where $-a\left(\frac{c_0}{a}\right)^{2/3} = 2c_0\left(\frac{a}{c_0}\right)^{1/3} + A$, giving $A = -3a^{1/3}c_0^{2/3}$.

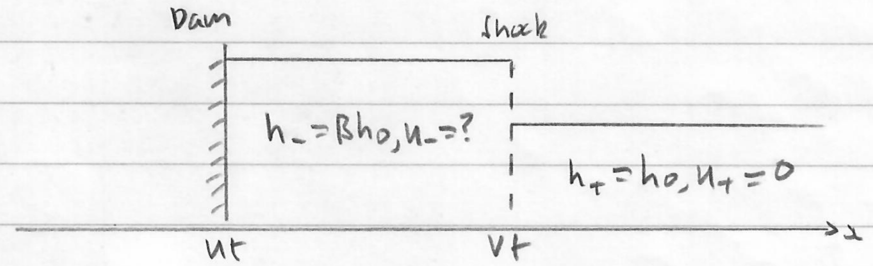
The element therefore reaches the origin at time $t = t_c$

[N5] where $0 = 2c_0 t_c^{1/3} + A \Rightarrow t_c^{1/3} = -A/2c_0 \Rightarrow t_c = \frac{27a}{8c_0}$.

~~(b)(ii) The BC on the dam says that the mass flux of water lost through the dam, i.e. $-\rho h(u-U)|_{x=Ut}$ where ρ is density, is proportional to the force exerted by the water on the dam, i.e. $\int_0^h P - P_{atm}\, dz = \int_0^h \rho g z\, dz \propto h^2$.~~

[S2]

(b)(i)



Rankine-Hugoniot conditions give

$$\beta h_0(u_- - V) = -h_0 V, \quad \beta h_0(u_- - V)^2 + \tfrac{1}{2}g\beta^2 h_0^2 = h_0 V^2 + \tfrac{1}{2}g h_0^2$$

$$\Rightarrow \quad u_- - V = -\frac{V}{\beta}, \quad \beta\left(-\frac{V}{\beta}\right)^2 + \tfrac{1}{2}g\beta^2 h_0 = V^2 + \tfrac{1}{2}g h_0$$

$$\Rightarrow \quad u_- = \left(1 - \tfrac{1}{\beta}\right)V, \quad V^2\left(1 - \tfrac{1}{\beta}\right) = \tfrac{1}{2}c_0^2(\beta^2 - 1)$$

Since we're assuming there is a shock $\beta \neq 1$ and $V > U > 0$,

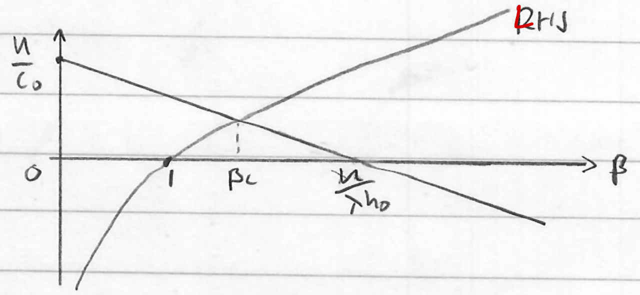
[B|S3] so taking +ve root gives $V = \frac{c_0}{2^{1/2}}\beta^{1/2}(\beta+1)^{1/2}$

The BC on the dam implies $u_- - U = -\lambda h_-$, giving

$$U - \lambda \beta h_0 = u_- = \frac{\beta - 1}{\beta} V = c_0 \frac{(\beta-1)(\beta+1)^{1/2}}{(2\beta)^{1/2}}$$

i.e. $$\frac{(\beta-1)(\beta+1)^{1/2}}{(2\beta)^{1/2}} = \frac{U}{c_0} - \frac{\lambda h_0}{c_0}\beta. \qquad (+)$$

Since the LHS ↗ with $\beta > 0$ and RHS ↘ with $\beta > 0$ as illustrated, $\exists!$ root $\beta = \beta_c$ to $(+)$ as illustrated.



[S|N3]

(b)(ii) Rate at which energy flows out of a stationary shock is

$$Q = \Big[\underbrace{\int_0^h \left(\tfrac{1}{2}\rho u^2 + \rho g z\right) u \, dz}_{\text{Rate at which KE \& PE ↗}} + \underbrace{\int_0^h (p - p_{atm}) u \, dz}_{\text{Rate at which work is done by pressure.}} \Big]_-^+$$

Since $p - p_{atm} = \rho g z$ in shallow water theory and $[hu]_-^+ = 0$,

$$Q = \left[\tfrac{1}{2}\rho h u^3 + \rho g h^2 u\right]_-^+ = \rho \underbrace{h_\pm u_\pm}_{\text{Conserved}} \left[\tfrac{1}{2}u^2 + gh\right]_-^+$$

Hence for our moving shock, replacing u with $u - V$,

$$Q = \rho h_-(u_- - V)\left[\tfrac{1}{2}(u-V)^2 + gh\right]_-^+ = \rho h_0(-V)\left(\tfrac{1}{2}V^2 + gh_0 - \tfrac{1}{2}\left(-\tfrac{V}{\beta}\right)^2 - g\beta h_0\right)$$

[S|N5] $$\Rightarrow Q = -\rho h_0 V\left(\tfrac{1}{2}\tfrac{gh_0}{2}\beta(\beta+1)\left(1 - \tfrac{1}{\beta^2}\right) + gh_0(1-\beta)\right) = \rho g h_0^2 V \underbrace{\frac{(1-\beta)^3}{4\beta}}_{q}$$

The shock can only be physical if energy lost as fluid crosses the shock (i.e. energy cannot be created), which requires $q < 0$. But $q < 0$

[N2] $\Rightarrow \beta_c > 1 \Rightarrow \frac{U}{\lambda h_0} > \beta_c > 1$ (from diagram) $\Rightarrow U > \lambda h_0$.

2014

3. A stream flowing over a horizontal surface is governed by the shallow water equations

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0,$$

where $u(x,t)$ is the fluid velocity in the x direction, $h(x,t)$ is the depth of the stream and g is the acceleration due to gravity.

(a) [6 marks] Show that the shallow water equations can be rewritten as

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)(u \pm 2c) = 0,$$

where $c = \sqrt{gh}$, and state the significance of the quantities $Q_\pm = u \pm 2c$ and the curves $\mathrm{d}x/\,\mathrm{d}t = u \pm c$.

At $t = 0$, a previously uniform stream $u = 1$, $h = 1$ is suddenly blocked by a dam at $x = 0$, so that $u(x,0) = h(x,0) = 1$, but $u(0,t) = 0$ for $t > 0$. You may assume that $g > 1$.

(b) [12 marks] Concentrate on $x > 0$, $t > 0$. By considering the characteristics that come from $\{t = 0,\ x > 0\}$, and $\{x = 0,\ t > 0\}$, deduce that the value of c immediately downstream of the dam is $c(0,t) = \sqrt{g} - \frac{1}{2}$. Also considering characteristics that come from $\{x = 0,\ t = 0\}$, deduce that the fluid velocity is given by

$$u(x,t) = \begin{cases} 0 & 0 < x < \left(\sqrt{g} - \frac{1}{2}\right)t, \\ \frac{2}{3}\left(\frac{x}{t} - \sqrt{g}\right) + \frac{1}{3} & \left(\sqrt{g} - \frac{1}{2}\right)t < x < \left(\sqrt{g} + 1\right)t, \\ 1 & \left(\sqrt{g} + 1\right)t < x, \end{cases}$$

and find a similar expression for $c(x,t)$.

(c) [4 marks] In $x < 0$, a shock propagates upstream with speed V, separating the uniform stream $u = 1$, $h = 1$ from a region of stationary water with depth $h_0 > 1$ adjacent to the dam. Show that the shock speed is

$$V = \frac{1}{h_0 - 1},$$

and that h_0 satisfies

$$2h_0 = g(h_0 + 1)(h_0 - 1)^2.$$

(d) [3 marks] Combining your solutions from parts (b) and (c), draw a sketch of the height profiles $h(x,t)$ (as a function of x) at $t = 1$ and at $t = 2$, on the same pair of axes.

[*You may assume the conditions for a shallow water shock moving with velocity* $-V$,

$$[h(u+V)]_-^+ = \left[h(u+V)^2 + \tfrac{1}{2}gh^2\right]_-^+ = 0,$$

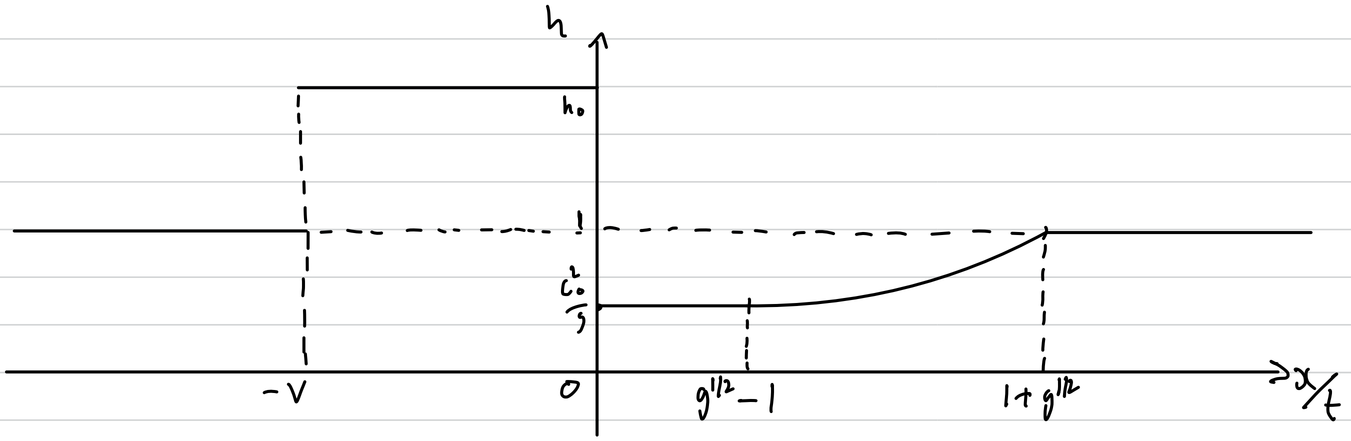
where $[\ ]_-^+$ *denotes the change in the quantity from one side of the shock to the other.*]

2014 Q3(d) Not quite sure how to combine previous parts? Was happy with all previous parts however.

Parts (b) & (c) $\Rightarrow$

$$h_0(x,t) = \begin{cases} 1 & \text{for } x < -Vt \\ h_0 & \text{for } -Vt < x < 0 \\ c_0^2/g & \text{for } 0 < x < (g^{1/2} - \frac{1}{2})t \\ \left(\frac{x}{t} + 2g^{1/2} - 1\right)^2/9g & \text{for } (g^{1/2} - \frac{1}{2})t < x < (1 + g^{1/2})t \\ g^{1/2} & \text{for } x > (1+g^{1/2})t \end{cases}$$

$$\left(c_0 = g^{1/2} - \frac{1}{2}\right)$$



2015

2. A barotropic gas with pressure-density relation $p = P(\rho)$ flows steadily past a thin symmetric wing, which occupies the region $|x| \leqslant a$ and has upper and lower surfaces given by $y = \pm f(x)$, where $f(\pm a) = 0$ and $|f| \ll a$. The flow is governed by Euler's equations for an inviscid compressible fluid,

$$\nabla \cdot (\rho \mathbf{u}) = 0, \qquad \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p.$$

Far upstream of the wing the flow is uniform, with speed U in the x direction, density ρ_0, pressure $p_0 = P(\rho_0)$, and sound speed given by $c_0^2 = P'(\rho_0)$.

(a) [10 marks] Assuming the flow is irrotational and writing $\mathbf{u} = U\hat{\mathbf{e}}_x + \nabla\phi$, find a linearised expression for the pressure perturbation in terms of the velocity potential, and show that $\phi(x, y)$ satisfies

$$(M^2 - 1)\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2},$$

where the Mach number M should be defined. Also derive the linearised boundary conditions

$$\frac{\partial \phi}{\partial y} = \pm U f'(x) \quad \text{on} \quad y = 0\pm,\ |x| \leqslant a,$$

where $y = 0\pm$ denotes the boundary approached from above and below, respectively.

Briefly describe the conditions that should be applied at infinity if $M < 1$, and explain why an appropriate condition in the case $M > 1$ is $\phi \to 0$ as $x \to -\infty$.

(b) [8 marks] Suppose now that $M > 1$. With the aid of a diagram to identify distinct regions of the flow, find the solution for $\phi(x, y)$.

(c) [7 marks] Suppose further that $f(x) = b(1 - x^2/a^2)$. Sketch the streamlines of the flow, and calculate the drag force D on the wing. Show that D is minimised (for $M > 1$) when $U = \sqrt{2}\, c_0$.

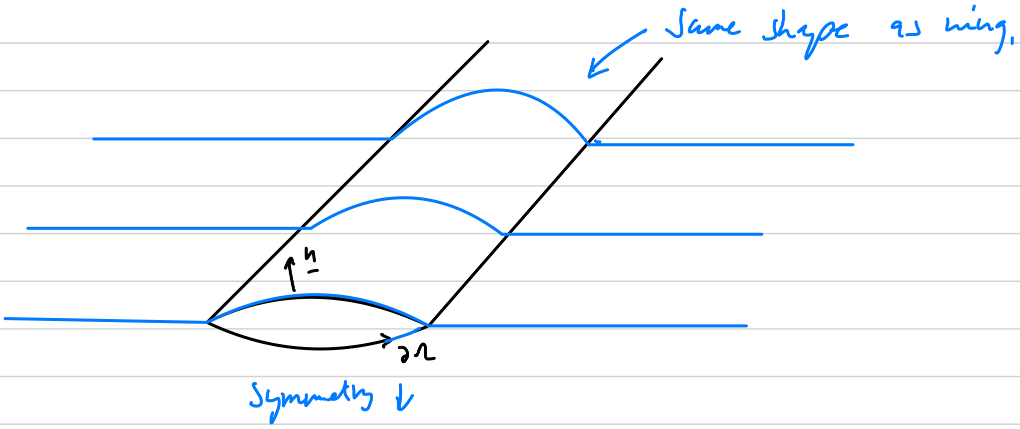
[*You may assume the expression*

$$D = \int_{-a}^{a} f'(x)\,[p(x, 0-) + p(x, 0+)]\,\mathrm{d}x,$$

for the drag force on the wing.]

Would it be possible to go over 2015 q2c please?

2015/Q2(c)



Force on wing $\underline{F} = \int_{\partial\Omega} -p\,\underline{n}\,ds$, $\underline{n} = \dfrac{(\mp f'(x), 1)}{\sqrt{1+f'(x)^2}}$

But given $D = \int_{-a}^{a} f'(x)\left(p(x,0-) + p(x,0+)\right) dx$ after linearization.

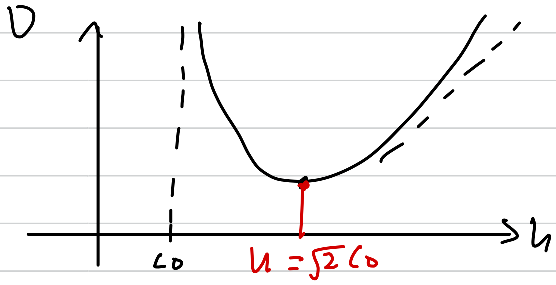
$$= -\rho_0 U \int_{-a}^{a} f'(x)\left(\phi_x(x,0-) + \phi_x(x,0+)\right) dx \qquad (p = -\rho_0 U \phi_x)$$

$$= +\frac{\rho_0 U^2}{\beta} \int_{-a}^{a} 2\, f'(x)^2\, dx \qquad (\text{by part (b)}) \quad (\beta = \sqrt{M^2-1})$$

$$= \frac{\rho_0 U^2}{\beta} \int_{-a}^{a} 2b^2 \left(-\frac{2x}{a^2}\right)^2 dx$$

$$= \frac{8\rho_0 U^2}{\beta}\, \frac{b^2}{a^4}\, \frac{2a^3}{3}$$

i.e. $D = \dfrac{16}{3}\,\dfrac{\rho_0 b^2 U^2}{a\sqrt{\frac{U^2}{c_0^2} - 1}}$



Want min of D for $U > c_0$.

$$\frac{dD}{dU^2} = \frac{16}{3}\,\frac{\rho_0 b^2}{a}\left[\frac{1}{\sqrt{\frac{U^2}{c_0^2}-1}} - \frac{U^2}{2\left(\frac{U^2}{c_0^2}-1\right)^{3/2} c_0^2}\right] = 0$$

$$\Rightarrow \quad 2\left(\frac{U^2}{c_0^2} - 1\right) = \frac{U^2}{c_0^2} \quad \Rightarrow \quad U^2 = 2c_0^2 \quad \text{or } U = \sqrt{2}\,c_0. \qquad \square$$

2015

3. A homentropic gas with pressure-density relation $p = k\rho^\gamma$ occupies a one-dimensional tube. The gas is initially at rest with density ρ_0 and pressure p_0, and lies to the right of an impermeable piston at $x = 0$. A measuring device is placed in the tube ahead of the piston, at $x = L$.

   (a) [3 marks] Briefly describe the physical principles underlying the Rankine-Hugoniot conditions for a shock moving with speed V,

$$[\rho(u-V)]_-^+ = \left[p + \rho(u-V)^2\right]_-^+ = \left[\frac{1}{2}(u-V)^2 + \frac{\gamma p}{(\gamma-1)\rho}\right]_-^+ = 0.$$

   (b) [10 marks] For $t > 0$ the piston is pushed forwards with constant speed U, causing a shockwave to travel ahead of it.

   Use the Rankine-Hugoniot conditions to show that the speed of the shock is given by

$$V = \frac{(\gamma+1)U}{4}\left[1 + \left(1 + \frac{16c_0^2}{(\gamma+1)^2U^2}\right)^{1/2}\right],$$

   where $c_0^2 = \gamma p_0/\rho_0$.

   Sketch the density $\rho(L,t)$ measured as a function of time at $x = L$, labelling all the important points on your graph. (Consider only $t < L/U$.)

   (c) [12 marks] Now suppose that instead of being pushed forwards, the piston is pulled backwards for $t > 0$, with constant speed U in the negative x direction.

   Assuming $U < 2c_0/(\gamma+1)$, and with the aid of a characteristic diagram to help find the solution $c(x,t)$, carefully sketch the sound speed $c(L,t)$ measured as a function of time at $x = L$, labelling all the important points on your graph.

   What is different if $U > 2c_0/(\gamma+1)$?

[*You may assume that the equations of one-dimensional homentropic gas dynamics,*

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$

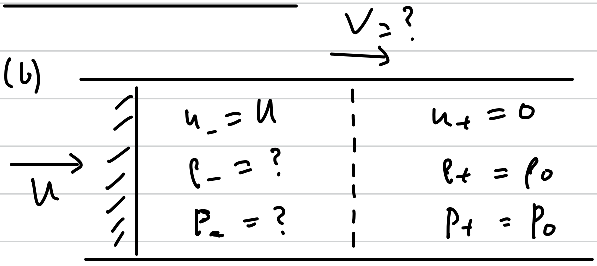
can be written as

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)\left(u \pm \frac{2c}{\gamma-1}\right) = 0,$$

where $c^2 = \gamma p/\rho$.]

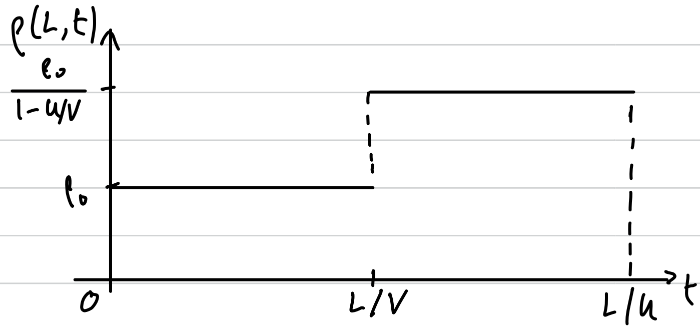
2015/ B5.4/ Q3

(b)

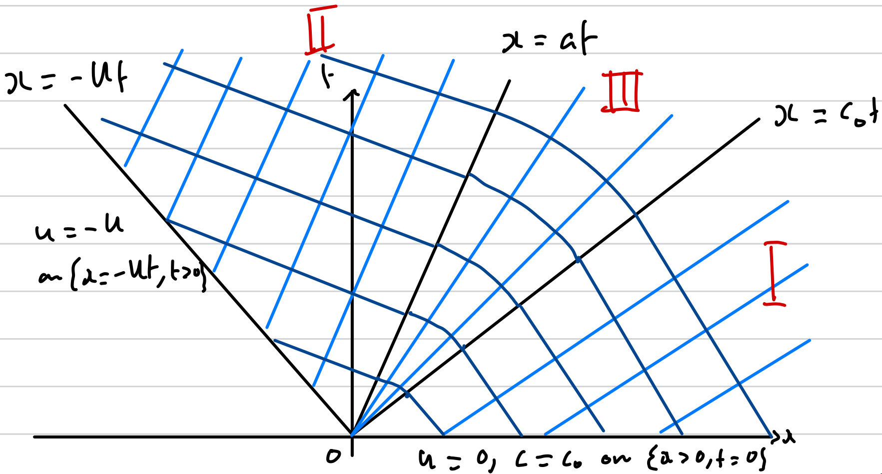


RHC $\Rightarrow$ $\rho_- = \rho_0 \dfrac{1}{1-U/V} > \rho_0 = \rho_+$

as $V > U$



(c)



$a = c_0 - \frac{\gamma+1}{2}U > 0$

Assume $U < \dfrac{2c_0}{\gamma+1}$

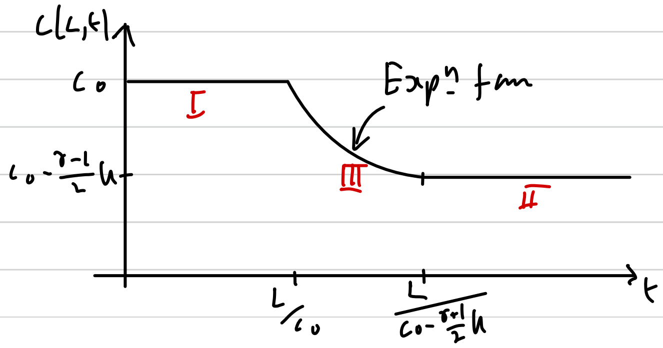
Know from lectures this means gas not left behind by piston (forming a vacuum), so that $u - \frac{2c}{\gamma-1} = -\frac{2c_0}{\gamma-1}$ everywhere (from − char^cs originating from $\{x>0, t=0\}$)

I : + char^cs also from $\{x>0, t=0\}$ $\Rightarrow$ $u = 0, \ c = c_0$ in $x > c_0 t, \ t>0$

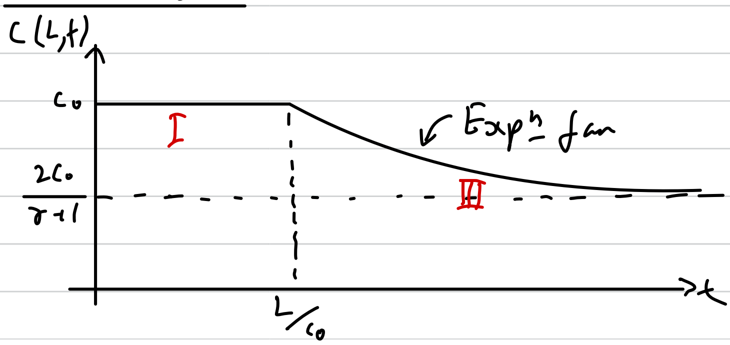
II : + char^cs from $\{x = -Ut, t>0\}$ $\Rightarrow$ $u = -U, \ c = c_0 - \frac{\gamma-1}{2}U$ in $-Ut < x < (c_0 - \frac{\gamma+1}{2}U)t$

III : + char^cs from $\{x=0, t=0\}$ $\Rightarrow$ $c = \frac{2c_0}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\frac{x}{t}$ for $(c_0 - \frac{\gamma+1}{2}U)t < x < c_0 t$

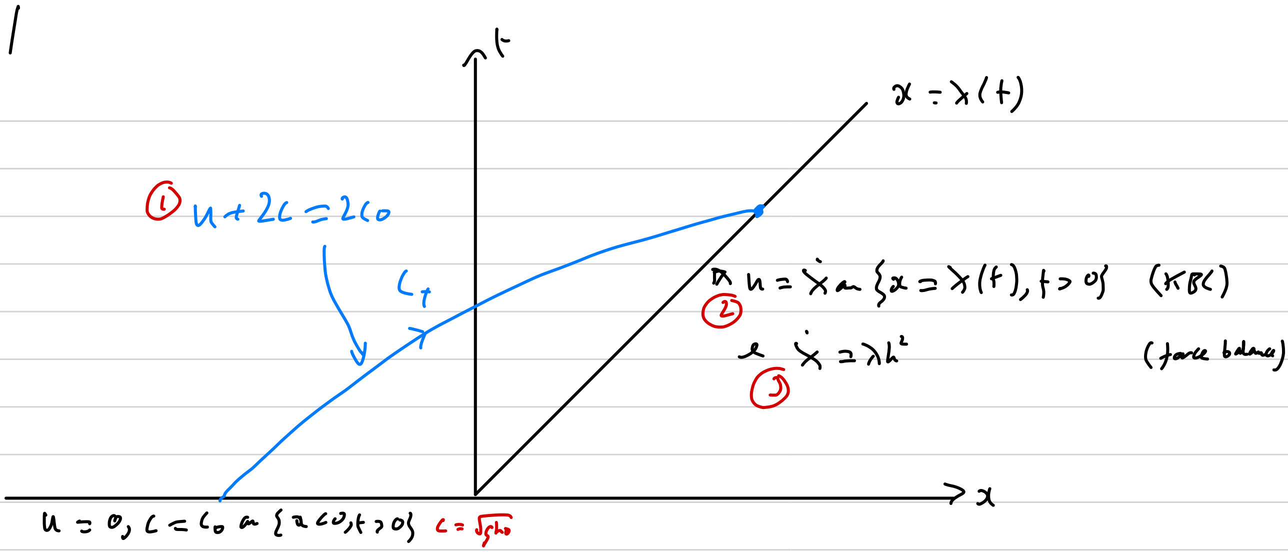
$U < 2c_0/\gamma+1$



$U > 2c_0/\gamma+1$



2015 / Q3 /



① - ③ with $c = \sqrt{gh}$ ⟹

$$\left.\begin{aligned} u + 2c &= 2c_0 \\ u &= \dot{X} \\ \dot{X} &= \lambda h^2 = \frac{\lambda c^4}{g^2} \end{aligned}\right\} \text{ on } x = X(t)$$

⟹ $u + 2c = 2c_0$ & $u = \dfrac{\lambda c^4}{g^2}$

i.e. u & c constant on piston, so $\dot{X} = \text{const.}$

Let $c(X, t) = \beta c_0$ where $\beta > 0$, then $u(X,t) = 2(c_0 - c) = 2(1-\beta)c_0$

and

$$2(1-\beta)c_0 = u(X,t) = \frac{\lambda\, c(X,t)^4}{g^2} = \frac{\lambda \beta^4 c_0^4}{g^2}$$

$$\Rightarrow \quad 1 - \beta = \frac{\lambda (g h_0)^{3/2}}{g^2} \beta^4 = \frac{\lambda h_0^{3/2}}{g^{1/2}} \beta^4$$

Now problem is identical to a piston withdrawal problem as $\dot{X} = 2(1-\beta)c_0$.