Problem Sheet 1

1. Exercise 1.6 on page 32 of the Lecture Notes.

2. Exercise 2.3 on page 56 of the Lecture Notes.

3. The Lotka-Volterra (predator-prey) system was studied in your Part A Differential Equations 1 course (see pages 38-39 of your lecture notes from last year). We write it as a chemical system

$$A \xrightarrow{k_1} 2A, \qquad B \xrightarrow{k_2} \emptyset, \qquad A + B \xrightarrow{k_3} 2B, \tag{$*$}$$

for two chemical species A ("prey") and B ("predator"). Its deterministic ODE model is

$$\frac{\mathrm{d}a}{\mathrm{d}t} = k_1 a - k_3\, a\, b, \qquad \frac{\mathrm{d}b}{\mathrm{d}t} = -k_2\, b + k_3\, a\, b, \tag{$\diamondsuit$}$$

where $a(t)$ and $b(t)$ are concentrations of A and B, respectively. Consider the (dimensionless) parameters $k_1 = k_2 = 10$ and $k_3 = 0.1$ and initial condition $a(0) = 50$ and $b(0) = 50$. For the stochastic case, consider (dimensionless) volume $\nu = 1$.

(a) Find critical points of ODEs ($\diamondsuit$). Investigate their stability, sketch the phase diagram and prove that the ODE system ($\diamondsuit$) has periodic solutions.

(b) Implement the Gillespie (a5)–(d5) for the chemical system ($*$). Plot the number of molecules of A and B as a function of time and compare your results with the solutions of ODEs ($\diamondsuit$). Plot both stochastic and deterministic trajectories $(A(t), B(t))$ and $(a(t), b(t))$ in the phase diagram as well. You should observe that, for a sufficiently long time, the deterministic and stochastic models give significantly different results. What types of long-time behaviour can the stochastic model have?

(c) Give an example of a chemical system which has the same deterministic description given by the ODEs ($\diamondsuit$), but its stochastic description (given by the Gillespie SSA) differs from the stochastic model corresponding to the chemical system ($*$).

(d) Consider the chemical system ($*$) together with two additional reactions

$$2A \xrightarrow{k_4} \emptyset, \qquad \emptyset \xrightarrow{k_5} A + B. \tag{$**$}$$

Then its deterministic ODE model is

$$\frac{\mathrm{d}a}{\mathrm{d}t} = k_1 a - k_3\, a\, b - 2k_4\, a^2 + k_5, \qquad \frac{\mathrm{d}b}{\mathrm{d}t} = -k_2\, b + k_3\, a\, b + k_5. \tag{$\heartsuit$}$$

Use dimensionless parameters $k_4 = 0.01$ and $k_5 = 1$. Show that the deterministic ODE model ($\heartsuit$) for the combined system ($*$)–($**$) does not have periodic solutions. What about its stochastic model? Does it oscillate? If yes, what is its period of oscillations?

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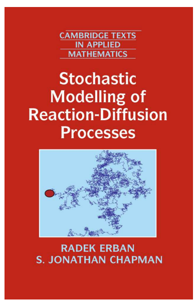
R. Erban and S.J. Chapman, "Stochastic Modelling of Reaction-Diffusion Processes", Cambridge Texts in Applied Mathematics, CUP (2020)



The online version is available to everyone at all times through SOLO. College libraries also have physical copies. Students wishing to further practice material covered in Chapters 1 and 2 can choose to solve any exercise in the Lecture Notes accompanying Chapter 1 (exercises on pages 29-32) and Chapter 2 (pages 55-58).