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Example 1 Consider the following differential operators on $\mathbb{R}^2$:

$A(\partial) = \partial_1 - \partial_2^2 + \partial_2$
$B(\partial) = \partial_1^2 - \partial_2^2 + \partial_1$
$C(\partial) = -\partial_1^2 - \partial_2^2 + \lambda^2$

where $\lambda > 0$. For each of them decide whether or not it is elliptic.

Example 2 Let $f \in \mathrm{L}^2(\mathbb{R}^2)$ and $\lambda > 0$. Prove that the PDE

$$-\Delta u + \lambda^2 u = f$$

has a unique solution $u \in \mathscr{S}'(\mathbb{R}^2)$ and that

$$\min\Big\{2\pi, \frac{2\pi}{\lambda^2}\Big\}\|f\|_2 \leq \|u\|_{\mathrm{H}^2} \leq \max\Big\{2\pi, \frac{2\pi}{\lambda^2}\Big\}\|f\|_2$$

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Example 3 Prove that the differential operator $-\Delta + \lambda^2$ ($\lambda > 0$) admits a unique fundamental solution $E \in \mathscr{S}'(\mathbb{R}^2)$. Use it to prove that the differential operator is hypoelliptic.

Example 4 Let Ω be an open nonempty subset of $\mathbb{R}^2$, $\lambda > 0$ and $g \in \mathsf{L}^2_{\mathrm{loc}}(\Omega)$. Show that all solutions to

$$-\Delta v + \lambda^2 v = g \ \text{ in } \ \mathscr{D}'(\mathbb{R}^2)$$

are regular distributions with $\partial^\alpha v \in \mathsf{L}^2_{\mathrm{loc}}(\Omega)$ for all multi-indices α of length $|\alpha| \leq 2$.