B4.4 Fourier Analysis Consultation Session 2 TT21

Example 1 Calculate the Fourier transform of the function $\mathrm{e}^{2x}H(-x)$, where H denotes the Heaviside function.
Let $\varepsilon > 0$ and consider $\frac{1}{x+\mathrm{i}\varepsilon}$. Explain why it is a tempered distribution on $\mathbb{R}$ and prove that the limit

$$\frac{1}{x+\mathrm{i}0} := \lim_{\varepsilon \searrow 0} \frac{1}{x+\mathrm{i}\varepsilon}$$

exists in $\mathscr{S}'(\mathbb{R})$. Calculate its Fourier transform.

Example 2 Let $p(x) \in \mathbb{C}[x]$ be a polynomial of one indeterminate.
Explain why $p(x)$ can be considered a tempered distribution on $\mathbb{R}$.
Calculate the Fourier transforms of $p(x)$ and of $p(x)H(x)$, where $H(x)$ is Heaviside's function.
Let $\xi_0 \in \mathbb{R}$. Find all $u \in \mathscr{S}'(\mathbb{R})$ satisfying $\mathrm{supp}(\widehat{u}) \subseteq \{\xi_0\}$.
Find all distributions $v \in \mathscr{S}'(\mathbb{R})$ satisfying $\mathrm{supp}(\widehat{v}) \subseteq \{0, 1\}$.

Example 3 Let H be the Heaviside function. Find the Fourier transform of $\mathrm{e}^{-2x}H(x)$. Next, find the Fourier transform of $\frac{1}{4+x^2}$.

Example 4 Calculate the Fourier transform of the function $f(x) = \mathrm{e}^{-|x|}$ on $\mathbb{R}$, and use it to establish the identity

$$\mathrm{e}^{-\lambda} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+\xi^2}\mathrm{e}^{\mathrm{i}\lambda\xi}\,\mathrm{d}\xi$$

for $\lambda \geq 0$. Next, use this together with $\frac{1}{1+\xi^2} = \int_0^\infty \mathrm{e}^{-(1+\xi^2)t}\,\mathrm{d}t$ to show that

$$\mathrm{e}^{-\lambda} = \int_0^\infty \frac{1}{\sqrt{\pi t}}\mathrm{e}^{-t-\frac{\lambda^2}{4t}}\,\mathrm{d}t \text{ for } \lambda \geq 0.$$

Calculate the Fourier transform of $\mathbb{R}^n \ni x \mapsto \mathrm{e}^{-t|x|}$, where $n \geq 2$ and $t > 0$ is a parameter.