B4.4 Fourier Analysis Consultation Session 3 TT21

Example 1 Show that $\big(\mathbf{1}_{(-1,1)} * \mathbf{1}_{(-1,1)}\big)(x) = \big(2 - |x|\big)^+$ for all $x \in \mathbb{R}^n$, where $t^+ := \max\{t, 0\}$. Hence, or otherwise, show that

$$\mathcal{F}_{x\to\xi}\left(\big(2 - |x|\big)^+\right) = 4\mathrm{sinc}^2\xi,$$

where sinc is sinus cardinalis. Use the Plancherel theorem to calculate the integral

$$\int_0^\infty \frac{\sin^2(ax)\sin^2(bx)}{x^4}\,\mathrm{d}x.$$

for all a, $b \in \mathbb{R}$.

Example 2 Prove the two inclusions

(i)

$$\bigcap_{s\in\mathbb{R}} \mathrm{H}^s(\mathbb{R}^n) \subset \mathsf{C}^\infty(\mathbb{R}^n)$$

(ii)

$$\mathscr{E}'(\mathbb{R}^n) \subset \bigcup_{s\in\mathbb{R}} \mathrm{H}^s(\mathbb{R}^n).$$

Explain why both inclusions are strict.

Example 3 Prove that for $s \in (0,1)$ and $f \in \mathsf{L}^2(\mathbb{R}^n)$ we have that $f \in \mathrm{H}^s(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x+y)-f(x)|^2}{|y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Hint: You may assume without proof that a continuous function $h\colon \mathbb{R}^n \to \mathbb{R}$ has the form $h(\xi) = h_0(|\xi|)$, where $h_0\colon [0,\infty) \to \mathbb{R}$ is a continuous function, provided $h(\theta\xi) = h(\xi)$ holds for all $\xi \in \mathbb{R}^n$ and $\theta \in \mathrm{SO}(n)$.