

B4.4 Fourier Analysis Consultation Session 4 TT21

Example 1 Suppose $u \in \mathcal{D}'(\mathbb{R})$ is 2π periodic. What is its *Fourier expansion* and in what sense does it hold? Briefly explain the connection to the Fourier transform of u .

Let $(\rho_\varepsilon)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}$. Show that $\rho_\varepsilon * u$ is 2π periodic and that the n -th Fourier coefficients satisfy $c_n(\rho_\varepsilon * u) \rightarrow c_n(u)$ as $\varepsilon \searrow 0$. Next prove that

$$c_n(\rho_\varepsilon * u) = c_n(u) \int_{-1}^1 \rho(y) e^{i\varepsilon ny} dy$$

and deduce that $|c_n(\rho_\varepsilon * u)| \leq |c_n(u)|$. Let $p(x) \in \mathbb{C}[x]$ be a polynomial in one indeterminate. Show that $p\left(\frac{d}{dx}\right)u$ is a 2π periodic tempered distribution and express its Fourier coefficients in terms of $c_n(u)$.

Example 2 Let u be a regular 2π periodic distribution on $\mathbb{R}$. Show that its n -th Fourier coefficient for $n \neq 0$ satisfies

$$c_n(u) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(u(x) - u\left(x + \frac{\pi}{n}\right) \right) e^{-inx} dx,$$

and deduce that $c_n(u) \rightarrow 0$ as $|n| \rightarrow \infty$.

Show that if v is a 2π periodic distribution of order 0, then its Fourier coefficients $c_n(v)$ are bounded:

$$\sup_{n \in \mathbb{Z}} |c_n(v)| < \infty.$$

Can this bound be strengthened to $c_n(v) \rightarrow 0$ as $|n| \rightarrow \infty$?

Example 3 Let $u \in H_{\text{loc}}^1(\mathbb{R})$ (so $u, u' \in L_{\text{loc}}^2(\mathbb{R})$) be 2π periodic. Show that its Fourier coefficients c_n satisfy

$$\sum_{n \in \mathbb{Z}} (1 + n^2) |c_n|^2 = k \int_0^{2\pi} (|u|^2 + |u'|^2) dx.$$

for some constant k that should be determined.

Example 4 Find $\mathcal{F}_{x \rightarrow \xi}(\sin x)$ and use it in conjunction with a differentiation rule to find $\mathcal{F}_{x \rightarrow \xi}(\text{sinc}(x))$. *HINT:* $\int_{\mathbb{R}} \text{sinc}(x) dx = \pi$. Next state a convolution rule and use it to find $\mathcal{F}_{x \rightarrow \xi}(\text{sinc}^2(x))$.

Calculate for each $x \in \mathbb{R}$ the infinite series

$$\sum_{n \in \mathbb{Z}} \text{sinc}^2(x + n\pi).$$