Fourier Analysis HT21

Problem Sheet 2

Problem 1. Prove that for every $t > 0$ and $\varphi \in \mathscr{S}(\mathbb{R})$ the identity

$$\int_{-t}^{t} \hat{\varphi}(\xi)\,\mathrm{d}\xi = 2\int_{-\infty}^{\infty} \varphi(x)\frac{\sin(tx)}{x}\,\mathrm{d}x$$

holds true. Deduce that

$$\lim_{t\to\infty} \frac{\sin(tx)}{x} = \pi\delta_0 \quad \text{in } \mathscr{S}'(\mathbb{R}),$$

where δ_0 is Dirac's delta-function concentrated at 0 on $\mathbb{R}$.
(*Hint: For instance use the product rule and the Fourier Inversion Formula in $\mathscr{S}'$ on the left-hand side of the identity.*)

Problem 2. Prove that $\mathscr{D}(\mathbb{R}^n)$ is $\mathscr{S}$ dense in $\mathscr{S}(\mathbb{R}^n)$: for each $\phi \in \mathscr{S}(\mathbb{R}^n)$ there exists a sequence (ϕ_j) in $\mathscr{D}(\mathbb{R}^n)$ so $\phi_j \to \phi$ in $\mathscr{S}(\mathbb{R}^n)$.

Problem 3. Define for $\alpha > 0$ the function $g(x) = \left(1 + |x|^2\right)^{-\frac{\alpha}{2}}$, $x \in \mathbb{R}^n$.
(a) Explain why $g \in \mathscr{S}'(\mathbb{R}^n)$. For which values of $\alpha > 0$ is g integrable over $\mathbb{R}^n$?
(b) Show that there exists a positive constant $c = c(\alpha)$ such that

$$g(x) = c\int_0^{\infty} t^{\frac{\alpha}{2}-1}\mathrm{e}^{-t}\mathrm{e}^{-t|x|^2}\,\mathrm{d}t$$

holds for all $x \in \mathbb{R}^n$.
(c) Using (b) show that the Fourier transform $\hat{g}$ is a positive and integrable function on $\mathbb{R}^n$.
(*The function $\hat{g}$ is called the Bessel kernel of order α.*)

Problem 4. For each of the following functions from $\mathbb{R}$ into $\mathbb{C}$ calculate its Fourier transform:

(i) $\cos$, $\sin$, $\cos^2$, $\cos^k$ for $k \in \mathbb{N}$.

(ii) sinc (sinus cardinalis, see the lecture notes Example 1.4).

(iii) H (Heaviside's function).

(iv) $xH(x) = x^+$, $|x|$, $\sin|x|$.

Deduce that

$$\mathcal{F}_{x\to\xi}\left(\mathrm{fp}\left(\tfrac{1}{x^2}\right)\right) = -\pi|\xi|, \quad (x,\,\xi \in \mathbb{R})$$

where the finite part distribution was defined on the B4.3 Problem Sheet 4.

Problem 5. Prove that

$$\int_0^\infty \frac{\sin(ax)\sin(bx)}{x^2}\,\mathrm{d}x = \frac{\pi}{2}\min\{a, b\}$$

holds for all a, $b > 0$.

Problem 6. (*Optional*) For each $\varepsilon > 0$ we put $H^\varepsilon(x) := \mathrm{e}^{-\varepsilon x}H(x)$, $x \in \mathbb{R}$, where H is Heaviside's function. Explain why $H^\varepsilon \in \mathscr{S}'(\mathbb{R})$ and show that

$$\frac{\mathrm{d}}{\mathrm{d}x}H^\varepsilon = -\varepsilon H^\varepsilon + \delta_0 \quad \text{in} \quad \mathscr{S}'(\mathbb{R}).$$

Show that

$$\frac{1}{\xi - \mathrm{i}\varepsilon} \underset{\varepsilon \searrow 0}{\longrightarrow} \mathrm{i}\widehat{H} \quad \text{in} \quad \mathscr{S}'(\mathbb{R}).$$

Using for instance Problem 4(iii) deduce the *Plemelj-Sokhotsky jump relation*:

$$\big(x + \mathrm{i}0\big)^{-1} - \big(x - \mathrm{i}0\big)^{-1} = -2\pi\mathrm{i}\delta_0,$$

where δ_0 is Dirac's delta-function on $\mathbb{R}$ concentrated at 0 and we define

$$\big(x \pm \mathrm{i}0\big)^{-1} = \lim_{\varepsilon \searrow 0}\big(x \pm \mathrm{i}\varepsilon\big)^{-1} \quad \text{in} \quad \mathscr{S}'(\mathbb{R}).$$