Fourier Analysis HT21

Problem Sheet 3

Problem 1. Let $f \in \mathrm{L}^p(\mathbb{R}^n)$ for some $p \in [1,\infty]$. Show that

$$g_j(\xi) = \int_{B_j(0)} f(x)\mathrm{e}^{-\mathrm{i}\xi\cdot x}\,\mathrm{d}x \quad (j \in \mathbb{N})$$

converge to $\widehat{f}$ in the sense of $\mathscr{S}'(\mathbb{R}^n)$ as $j \to \infty$. Hence find the limit of

$$\int_{-j}^{j} x^k \mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x$$

in $\mathscr{S}'(\mathbb{R})$ as $j \to \infty$ for each $k \in \mathbb{N}_0$.

Problem 2. Let $p(x) \in \mathbb{C}[x]$ be a polynomial in n variables. Find the Fourier transform $\widehat{p}$.
Find all distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ satisfying $\mathrm{supp}(\widehat{u}) \subseteq \{0\}$.
(*Hint:* Use a theorem about distributions with support in $\{0\}$ from B4.3.)

Problem 3. Prove that e^x is not in $\mathscr{S}'(\mathbb{R})$ but that $\mathrm{e}^x\mathrm{e}^{\mathrm{i}\mathrm{e}^x}$ is in $\mathscr{S}'(\mathbb{R})$.

Problem 4.
(a) Recall that we defined homogeneity of distributions $u \in \mathscr{D}'(\mathbb{R}^n)$ in B4.3 by declaring that u is *homogeneous of degree* α, where $\alpha \in \mathbb{R}$, if $d_r u = r^\alpha u$ holds for each $r > 0$.
Show that $u \in \mathscr{S}'(\mathbb{R}^n)$ is homogeneous of degree α if and only if $\widehat{u}$ is homogeneous of degree $-n-\alpha$.
(b) A distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ is *radial* if $\theta_* u = u$ for each $\theta \in \mathrm{O}(n)$.
Show that $u \in \mathscr{S}'(\mathbb{R}^n)$ is radial if and only if $\widehat{u}$ is radial. Next, show that if u is a regular distribution, then u is radial if and only if $u(x) = f(|x|)$ a.e., where $f\colon \mathbb{R} \to \mathbb{C}$ is a univariate function. (*You may use elementary properties of orthogonal matrices without proof.*)
(c) Use the Fourier transform to show that there exists a constant $c \in \mathbb{R}$ (that you should not find) so that $c/|x|$ is a fundamental solution to the Laplacian on $\mathbb{R}^3$. (*Hint:* $|\xi|^{-2} \in \mathrm{L}^1(\mathbb{R}^3) + \mathrm{L}^2(\mathbb{R}^3)$.)

Problem 5. [The Wiener algebra $\mathcal{F}(\mathrm{L}^1)$ is strictly smaller than C_0]
(a) Prove that for $0 < s < t$ we have

$$\left| \int_s^t \frac{\sin(\xi)}{\xi}\,\mathrm{d}\xi \right| \le 4.$$

(b) Show that if $f \in \mathrm{L}^1(\mathbb{R})$ is an odd distribution (so $\widetilde{f} = -f$), then

$$\left| \int_s^t \frac{\widehat{f}(\xi)}{\xi}\,\mathrm{d}\xi \right| \le 4\|f\|_1.$$

(c) Let $g \in \mathrm{C}_0(\mathbb{R})$ be an odd function so that $g(\xi) = 1/\log(\xi)$ for $\xi \geq 2$. Show that there does not exist an integrable function whose Fourier transform is g.

Problem 6. [S. Bernstein's inequality]
Let $f \in \mathrm{L}^\infty(\mathbb{R}^n)$ and assume that $\widehat{f}$ is supported in the closed ball $\overline{B_r(0)}$. Prove that $f \in \mathrm{C}^\infty(\mathbb{R}^n)$ and that for each multi-index $\alpha \in \mathbb{N}_0^n$ there exists a constant $c = c(\alpha, n) \geq 0$ such that

$$\|\partial^\alpha f\|_\infty \leq c r^{|\alpha|} \|f\|_\infty.$$

(*Hint:* Let $h \in \mathscr{S}(\mathbb{R}^n)$ be chosen so $\widehat{h} = 1$ on $B_1(0)$ and $\widehat{h} = 0$ off $B_2(0)$. Note that $f = f * h_{1/r}$.)