

Problem Sheet 4

Problem 1. [A theorem of Paley and Wiener.]

(a) Let $f \in L^2(\mathbb{R})$ and assume that $f(x) = 0$ for a. e. $x > 0$. Prove that

$$F(\zeta) := \int_{-\infty}^0 f(x) e^{-i\zeta x} dx$$

is a well-defined and holomorphic function for all $\zeta \in \mathbb{C}$ with $\text{Im}(\zeta) > 0$.

Show that F satisfies the bound

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\xi + i\eta)|^2 d\xi = \int_{-\infty}^0 |f(x)|^2 e^{2\eta x} dx \leq \|f\|_2^2$$

for all $\eta > 0$. Next, show that

$$F(\cdot + i\eta) \rightarrow \widehat{f} \text{ in } L^2(\mathbb{R}) \text{ as } \eta \searrow 0.$$

(b) (*Optional.*) Assume that $F: H \rightarrow \mathbb{C}$ is a holomorphic function defined on the upper half-plane $H = \{\zeta \in \mathbb{C} : \text{Im}(\zeta) > 0\}$ that satisfies the bound

$$\sup_{\eta > 0} \int_{-\infty}^{\infty} |F(\xi + i\eta)|^2 d\xi < \infty.$$

Prove that $F(\cdot + i\eta)$ converges in $L^2(\mathbb{R})$ as $\eta \searrow 0$ to the Fourier transform \widehat{f} of a function $f \in L^2(\mathbb{R})$ vanishing on $(0, \infty)$.

Problem 2.

(a) Show that the Fourier transform of the function

$$f(x) = (1 - |x|)^+ = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

is $\text{sinc}^2(\xi/2)$. (See Example 1.4 in the lecture notes for the definition of sinus cardinalis. However, the calculation is probably easiest via the differentiation rule and double-angle formula.) Explain why the Poisson summation formula applies to f and use it to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+x)^2} = \frac{\pi^2}{\sin^2(\pi x)} \quad (1)$$

holds for all $x \in \mathbb{R} \setminus \mathbb{Z}$.

(b) For instance by use of (a) show that

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{n+x} = \pi \cot(\pi x) \quad (2)$$

holds for all $x \in \mathbb{R} \setminus \mathbb{Z}$. (Hint: Consider the identity first for $x \in (0, 1)$. The difference of the two sides is a distribution in $\mathcal{D}'(0, 1)$ —what is its derivative?)

(c) Explain why the identity (2) can be extended to hold in $\mathbb{C} \setminus \mathbb{Z}$:

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{n+z} = \pi \cot(\pi z)$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$. Using this formula deduce *Lipschitz's formula*:

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z}$$

holds for all $z \in \mathbb{C}$ with $\text{Im}(z) > 0$ and $k \in \mathbb{N}$, $k \geq 2$.

Problem 3. (Optional.) Consider the 2π -periodic function $g: \mathbb{R} \rightarrow \mathbb{C}$ given by

$$g(x) = \frac{\pi}{\sin \pi \alpha} e^{i(\pi-x)\alpha}$$

for $x \in (0, 2\pi]$, where $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Calculate its Fourier series and use Parseval's identity for Fourier series to recover (1). In what sense does the Fourier series for g converge and could we use it to recover (2)?

Problem 4. For each $\varphi \in \mathcal{S}(\mathbb{R})$ we define its *periodisation* as

$$P\varphi(x) = \sum_{k \in \mathbb{Z}} \varphi(x - 2\pi k), \quad x \in \mathbb{R}.$$

(a) Check that $P\varphi$ is a 2π periodic C^∞ function, and show that

$$P\varphi(x) = \sum_{k \in \mathbb{Z}} \frac{1}{2\pi} \widehat{\varphi}(k) e^{ikx}$$

holds for all $x \in \mathbb{R}$. Deduce Poisson's summation formula:

$$2\pi \sum_{k \in \mathbb{Z}} \varphi(2\pi k) = \sum_{k \in \mathbb{Z}} \widehat{\varphi}(k).$$

(c) Show that

$$\sum_{k \in \mathbb{Z}} e^{-4\pi^2 t k^2} = \frac{1}{\sqrt{4\pi t}} \sum_{k \in \mathbb{Z}} e^{-\frac{k^2}{4t}}$$

holds for all $t > 0$.

Problem 5. [Fast convergence of Riemann sums for test functions.]

Let $\varphi \in \mathcal{D}(\mathbb{R})$ and $m \in \mathbb{N}$. Show that

$$\int_{\mathbb{R}} \varphi(x) dx = \frac{1}{N} \sum_{n \in \mathbb{Z}} \varphi\left(\frac{n}{N}\right) + O\left(\frac{1}{N^m}\right) \text{ as } N \rightarrow \infty.$$

Here the implied constant may of course depend on φ and m .

Problem 6.

(a) Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a 2π -periodic L^1_{loc} function. Show that for $n \in \mathbb{Z} \setminus \{0\}$ its n -th Fourier coefficient c_n satisfies

$$c_n = \frac{1}{4\pi} \int_{-\pi}^{\pi} (f(x) - f(x + \frac{\pi}{n})) e^{-inx} dx.$$

Deduce that $c_n \rightarrow 0$ as $|n| \rightarrow \infty$. This result is also called the *Riemann–Lebesgue lemma*.

(b) Show that the Fourier coefficients of a continuous 2π -periodic function can tend to 0 arbitrarily slowly by proving that for every sequence (t_n) of positive numbers converging to 0, there exists a continuous function f whose Fourier coefficients satisfy $|c_n| \geq t_n$ for infinitely many values of n . (*Hint: Choose a subsequence (t_{n_k}) so $\sum_{k=1}^{\infty} t_{n_k} < \infty$.*)

(c) Assume that $f: \mathbb{R} \rightarrow \mathbb{C}$ is 2π -periodic and α -Hölder continuous for some $\alpha \in (0, 1]$:

$$|f(x) - f(y)| \leq c|x - y|^\alpha$$

holds for all $x, y \in \mathbb{R}$, where $c \geq 0$ is a constant. Using (a) prove that the Fourier coefficients c_n satisfy

$$c_n = O(|n|^{-\alpha}) \text{ as } |n| \rightarrow \infty.$$

(d) Assume that $f: \mathbb{R} \rightarrow \mathbb{C}$ is a 2π -periodic H^1_{loc} function (so both f and f' belong to $L^2_{\text{loc}}(\mathbb{R})$). Prove that the Fourier series for f converges absolutely and uniformly.