B4.4 Fourier Analysis HT21

Lecture 5: Tempered distributions and the adjoint identity scheme revisited

1. Definition of tempered distributions
2. Comparison of the different classes of distributions
3. Examples: tempered L^p functions and tempered measures
4. The boundedness property of tempered distributions
5. The adjoint identity scheme in the tempered context

The material corresponds to pp. 20–25 in the lecture notes and should be covered in Week 3.

An adjoint identity for the Fourier transform

We have proved that the Fourier transform is a bijective $\mathscr{S}$ continuous linear map $\mathcal{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ with inverse $\mathcal{F}^{-1} = (2\pi)^{-n}\widetilde{\mathcal{F}}$. In view of this the product rule, when restricted to Schwartz test functions, becomes an adjoint identity:

$$\int_{\mathbb{R}^n} \mathcal{F}(\phi)\psi \, \mathrm{d}x = \int_{\mathbb{R}^n} \phi\mathcal{F}(\psi) \, \mathrm{d}x$$

holds for all ϕ, $\psi \in \mathscr{S}(\mathbb{R}^n)$. We shall take advantage of this and extend the Fourier transform, in a consistent manner, to a large class of distributions. This is the motivation for introducing the class of Schwartz test function.

Definition of tempered distributions

Definition A functional $u\colon \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ is a *tempered distribution on* $\mathbb{R}^n$ if

(i) u is linear,

(ii) u is $\mathscr{S}$ continuous: if $\phi_j \to \phi$ in $\mathscr{S}(\mathbb{R}^n)$, then $u(\phi_j) \to u(\phi)$.

The set of all tempered distributions on $\mathbb{R}^n$ is denoted by $\mathscr{S}'(\mathbb{R}^n)$.

Remarks

- When $u\colon \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ is linear, then (ii) holds provided u is $\mathscr{S}$ continuous at 0.
- Under the usual definitions of vector space operations it is clear that $\mathscr{S}'(\mathbb{R}^n)$ becomes a vector space over $\mathbb{C}$.
- We shall also use the bracket notation for tempered distributions and often write $\langle u, \phi \rangle$ instead of $u(\phi)$.

Relation to other classes of distributions from B4.3

We have introduced the classes of distributions $\mathscr{D}'(\mathbb{R}^n)$ and $\mathscr{E}'(\mathbb{R}^n)$ on $\mathbb{R}^n$. How are these classes related to the tempered distributions? – First note that

$$\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset \mathrm{C}^\infty(\mathbb{R}^n)$$

where the two inclusions are strict. We claim that

$$\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$$

and that the two inclusions are strict too. First, one may wonder what it means. The argument below will however make that clear.
Let $u \in \mathscr{S}'(\mathbb{R}^n)$. Then its restriction $u|_{\mathscr{D}(\mathbb{R}^n)}$ to the subspace $\mathscr{D}(\mathbb{R}^n)$ is clearly still linear. If $\phi_j \to 0$ in $\mathscr{D}(\mathbb{R}^n)$, then as we have seen before the convergence also takes place in the $\mathscr{S}$ sense, so by assumption

$$\langle u|_{\mathscr{D}(\mathbb{R}^n)}, \phi_j \rangle = \langle u, \phi_j \rangle \to 0,$$

hence the restriction $u|_{\mathscr{D}(\mathbb{R}^n)} \in \mathscr{D}'(\mathbb{R}^n)$. It is in this sense we intend the inclusion above. We also emphasize that the restriction $u|_{\mathscr{D}(\mathbb{R}^n)}$ uniquely determines $u \in \mathscr{S}'(\mathbb{R}^n)$ because $\mathscr{D}(\mathbb{R}^n)$ is $\mathscr{S}$ dense in $\mathscr{S}(\mathbb{R}^n)$.

Relation to other classes of distributions

The inclusion is strict since $\mathrm{e}^{|x|^2} \in \mathscr{D}'(\mathbb{R}^n) \setminus \mathscr{S}'(\mathbb{R}^n)$: if $u \in \mathscr{S}'(\mathbb{R}^n)$ and $\langle u, \phi \rangle = \int_{\mathbb{R}^n} \phi(x) \mathrm{e}^{|x|^2} \, \mathrm{d}x$ for $\phi \in \mathscr{D}(\mathbb{R}^n)$, then approximating $\mathrm{e}^{-|x|^2} \in \mathscr{S}(\mathbb{R}^n)$ by $\phi_j \in \mathscr{D}(\mathbb{R}^n)$ in the $\mathscr{S}$ sense we get a constradiction,

$$\langle u, \mathrm{e}^{-|\cdot|^2} \rangle = \lim_{j \to \infty} \langle u, \phi_j \rangle = \lim_{j \to \infty} \int_{\mathbb{R}^n} \mathrm{e}^{|x|^2} \phi_j(x) \, \mathrm{d}x = \infty.$$

We turn to the compactly supported distributions and let $u \in \mathscr{E}'(\mathbb{R}^n)$. We recall from B4.3 that u admits a unique extension, denoted u again, to a linear functional on $\mathrm{C}^\infty(\mathbb{R}^n)$ with the property that for each compact neighbourhood K of the support $\mathrm{supp}(u)$ there exist constants $c = c_K \geq 0$, $m = m_K \in \mathbb{N}_0$ so

$$\bigl| \langle u, \phi \rangle \bigr| \leq c \sum_{|\alpha| \leq m} \sup_K \bigl| \partial^\alpha \phi \bigr|$$

holds for all $\phi \in \mathrm{C}^\infty(\mathbb{R}^n)$.

Relation to other classes of distributions

Clearly the restriction $u|_{\mathscr{S}(\mathbb{R}^n)}$ remains linear and if $\phi_j \to 0$ in $\mathscr{S}(\mathbb{R}^n)$, then

$$\begin{aligned} \left|\langle u|_{\mathscr{S}(\mathbb{R}^n)}, \phi_j\rangle\right| = \left|\langle u, \phi_j\rangle\right| &\leq c \sum_{|\alpha|\leq m} \sup_K \left|\partial^\alpha \phi_j\right| \\ &\leq c\Bigl(\sum_{|\alpha|\leq m} 1\Bigr) \overline{S}_{0,m}(\phi_j) \to 0, \end{aligned}$$

so $u|_{\mathscr{S}(\mathbb{R}^n)} \in \mathscr{S}'(\mathbb{R}^n)$, and it is in this sense the inclusion should be understood. Again, the inclusion is strict since $\mathrm{e}^{-|x|^2} \in \mathscr{S}'(\mathbb{R}^n) \setminus \mathscr{E}'(\mathbb{R}^n)$.

As already indicated above, we shall omit writing *restrictions* here, and for instance simply write that $u \in \mathscr{S}'(\mathbb{R}^n)$ when we actually mean $u|_{\mathscr{S}(\mathbb{R}^n)} \in \mathscr{S}'(\mathbb{R}^n)$.

Example 1. Let $f \in \mathrm{L}^p(\mathbb{R}^n)$, where $p \in [1, \infty]$. Define

$$T_f(\phi) = \int_{\mathbb{R}^n} f \phi \, \mathrm{d}x, \ \phi \in \mathscr{S}(\mathbb{R}^n).$$

Then T_f is well-defined and linear. By Hölder's inequality and the inclusion $\mathscr{S}(\mathbb{R}^n) \subset \mathrm{L}^q(\mathbb{R}^n)$, where q is the Hölder conjugate exponent to p, we get

$$|T_f(\phi)| \leq \|f\|_p \|\phi\|_q \leq c(n, q) \|f\|_p \overline{S}_{n+1,0}(\phi).$$

Therefore T_f is also $\mathscr{S}$ continuous, so $T_f \in \mathscr{S}'(\mathbb{R}^n)$. As observed before T_f, or its restriction to $\mathscr{D}(\mathbb{R}^n)$, is then a distribution in $\mathscr{D}'(\mathbb{R}^n)$ too, and so f is uniquely determined (by the fundamental lemma of the calculus of vairations). We shall therefore also identify T_f and f for tempered distributions, and simply write $T_f = f$, where it is then clear from context or else must be explicitly mentioned in what capacity f is considered.

Example 2. Let μ be a finite Borel measure on $\mathbb{R}^n$. Define

$$T_\mu(\phi) = \int_{\mathbb{R}^n} \phi \, \mathrm{d}\mu, \; \phi \in \mathscr{S}(\mathbb{R}^n).$$

Then T_μ is well-defined and linear. Since also $|T_\mu(\phi)| \leq \mu(\mathbb{R}^n) S_{0,0}(\phi)$ it follows that $T_\mu \in \mathscr{S}'(\mathbb{R}^n)$. As in the previous example T_μ, or its restriction to $\mathscr{D}(\mathbb{R}^n)$ is a distribution in $\mathscr{D}'(\mathbb{R}^n)$ and so μ is uniquely determined by T_μ. We therefore identify T_μ with μ and write simply $T_\mu = \mu$ also in this case. In particular note that the Dirac delta function δ_a also can be viewed as a tempered distribution.

Example 3. Functions in $\mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}^n)$ and locally finite Borel measures do not in general define tempered distributions. As we have seen, $\mathrm{e}^{|x|^2} \in \mathsf{L}^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ does not define a tempered distribution. In order to be a tempered distribution a function should not grow too fast at infinity. This is vague and, as it turns out, it has to be. For example you will show on problem sheet 3 that $\mathrm{e}^x \notin \mathscr{S}'(\mathbb{R})$, while $\mathrm{e}^{x+\mathrm{e}^{\mathrm{i}x}} \in \mathscr{S}'(\mathbb{R})$.

Tempered L^p functions and measures

In the context of the distributions in $\mathscr{D}'$ the *regular distributions* were those corresponding to $\mathsf{L}^1_{\mathrm{loc}}$ functions. The corresponding notion of *regular tempered distribution* is the notion of a *tempered* L^1 *function*.

Definition Let $p \in [1, \infty]$. A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is (a representative for) a *tempered* L^p *function* if there exists $m \in \mathbb{N}_0$ so

$$\frac{f(x)}{\left(1+|x|^2\right)^{\frac{m}{2}}} \in \mathsf{L}^p(\mathbb{R}^n). \tag{1}$$

A Borel measure μ on $\mathbb{R}^n$ is a *tempered measure* if for some $m \in \mathbb{N}_0$ we have

$$\int_{\mathbb{R}^n} \frac{\mathrm{d}\mu(x)}{\left(1+|x|^2\right)^{\frac{m}{2}}} < \infty. \tag{2}$$

Tempered L^p functions and tempered measures are tempered distributions: Assume f is a tempered L^p function and μ a tempered measure, say (1) and (2) hold. Then if $\phi \in \mathscr{S}(\mathbb{R}^n)$ we define

$$\langle T_f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)\,\mathrm{d}x \ \text{ and } \ \langle T_\mu, \phi \rangle = \int_{\mathbb{R}^n} \phi\,\mathrm{d}\mu.$$

We claim they are well-defined tempered distributions. To see that T_f is, use Hölder's inequality,

$$\begin{aligned} \bigl|\langle T_f, \phi \rangle\bigr| \leq \int_{\mathbb{R}^n} \bigl|f\phi\bigr|\,\mathrm{d}x \ &\leq \ \left\| \frac{f(\cdot)}{\left(1+|\cdot|^2\right)^{\frac{m}{2}}} \right\|_p \left\| \left(1+|\cdot|^2\right)^{\frac{m}{2}} \phi \right\|_q \\ &\leq \ c \left\| \frac{f(\cdot)}{\left(1+|\cdot|^2\right)^{\frac{m}{2}}} \right\|_p \overline{S}_{n+1+m,0}(\phi) \end{aligned}$$

so T_f is well-defined and hence linear. It also follows from the bound that it is $\mathscr{S}$ continuous. The proof for T_μ is easier and left as an exercise.

Tempered L^p functions and measures

As we have seen that $\mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$ also T_f, $T_\mu \in \mathscr{D}'(\mathbb{R}^n)$ and so we may also in the tempered context identify T_f with f and T_μ with μ. Henceforth we therefore also write

$$T_f = f$$

for tempered L^p functions and

$$T_\mu = \mu$$

for tempered measures.

The boundedness property of tempered distributions

Proposition Let $u \colon \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ be linear. Then u is $\mathscr{S}$ continuous if and only if there exist constants $c \geq 0$, k, $l \in \mathbb{N}_0$ so

$$\left| \langle u, \phi \rangle \right| \leq c \overline{S}_{k,l}(\phi)$$

holds for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

Note that the boundedness property implies that tempered distributions always have a finite order (the order is at most l if the above bound holds for u).

The boundedness property of tempered distributions

Proof. It is clear that the bound together with linearity implies $\mathscr{S}$ continuity. So we focus on the opposite direction and assume that u is $\mathscr{S}$ continuous. The proof goes by contradiction: assume that the boundedness property fails. Then for all $c = k = l = j \in \mathbb{N}$ there exists $\phi_j \in \mathscr{S}(\mathbb{R}^n)$ so

$$\big|\langle u, \phi_j \rangle\big| > j\overline{S}_{j,j}(\phi_j).$$

Then clearly $\phi_j \neq 0$, so $\overline{S}_{j,j}(\phi_j) > 0$ and we may define

$$\psi_j = \frac{\phi_j}{j\overline{S}_{j,j}(\phi_j)} \in \mathscr{S}(\mathbb{R}^n).$$

Fix α, $\beta \in \mathbb{N}_0^n$. Then for $j > |\alpha| + |\beta|$ we have $S_{\alpha,\beta}(\psi_j) < 1/j$, so by arbitrariness of α, β we have shown that $\psi_j \to 0$ in $\mathscr{S}(\mathbb{R}^n)$. Consequently we must by $\mathscr{S}$ continuity have $\langle u, \psi_j \rangle \to 0$. But this is impossible because we also have $|\langle u, \psi_j \rangle| > 1$. $\square$

Convergence of tempered distributions

Definition For a sequence (u_j) in $\mathscr{S}'(\mathbb{R}^n)$ and $u \in \mathscr{S}'(\mathbb{R}^n)$ we write $u_j \to u$ in $\mathscr{S}'(\mathbb{R}^n)$ if $\langle u_j, \phi \rangle \to \langle u, \phi \rangle$ holds for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

Because $\mathscr{D}(\mathbb{R}^n)$ is a proper subspace of $\mathscr{S}(\mathbb{R}^n)$ this mode of convergence is clearly strictly stronger than convergence in $\mathscr{D}'(\mathbb{R}^n)$.

Example Find the limits in the sense of tempered distributions of

(i) $\big(\sin(jx)\big)$ as $j \to \infty$,

(ii) $\big(\rho_\varepsilon\big)$ as $\varepsilon \searrow 0$.

(i): We know from B4.3 that $\sin(jx) \to 0$ in $\mathscr{D}'(\mathbb{R}^n)$. Because $\mathscr{D}(\mathbb{R})$ is $\mathscr{S}$ dense in $\mathscr{S}(\mathbb{R})$, given $\phi \in \mathscr{S}(\mathbb{R})$ and $\varepsilon > 0$ we can find $\psi \in \mathscr{D}(\mathbb{R})$ with $\overline{S}_{2,0}(\phi - \psi) < \varepsilon$.

Convergence of tempered distributions

Now

$$\begin{aligned}
\left|\int_{\mathbb{R}} \sin(jx)\phi(x)\,\mathrm{d}x\right| &\le \left|\int_{\mathbb{R}} \sin(jx)\psi(x)\,\mathrm{d}x\right| + \int_{\mathbb{R}} \left|\sin(jx)\right|\left|\phi(x)-\psi(x)\right|\,\mathrm{d}x \\
&\le \left|\int_{\mathbb{R}} \sin(jx)\psi(x)\,\mathrm{d}x\right| \\
&\quad + \int_{\mathbb{R}} \frac{\mathrm{d}x}{1+x^2} \sup_{x\in\mathbb{R}}\Big((1+x^2)\left|\phi(x)-\psi(x)\right|\Big) \\
&\le \left|\int_{\mathbb{R}} \sin(jx)\psi(x)\,\mathrm{d}x\right| + 2\pi\overline{S}_{2,0}(\phi-\psi) \\
&\le \left|\int_{\mathbb{R}} \sin(jx)\psi(x)\,\mathrm{d}x\right| + 2\pi\varepsilon.
\end{aligned}$$

It follows that $\sin(jx) \to 0$ in $\mathscr{S}'(\mathbb{R})$ as $j \to \infty$.

Convergence of tempered distributions

We could of course also have proceeded exactly as we did in B4.3, simply replacing the $\mathscr{D}$ test functions by Schwartz test functions throughout. However we wanted to point out that many results from B4.3 can also be transferred without much effort using $\mathscr{S}$ density of $\mathscr{D}(\mathbb{R}^n)$ in $\mathscr{S}(\mathbb{R}^n)$.

(ii): $\rho_\varepsilon \to \delta_0$ in $\mathscr{S}'(\mathbb{R})$ as $\varepsilon \searrow 0$.

Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Then by uniform convergence we get since $\operatorname{supp}(\rho) = \overline{B_1(0)}$ has finite measure:

$$\langle \rho_\varepsilon, \phi \rangle = \int_{\mathbb{R}^n} \rho(x)\phi(\varepsilon x)\,\mathrm{d}x \to \phi(0)$$

as $\varepsilon \searrow 0$.

The adjoint identity scheme in the tempered context

The procedure is as in B4.3 and the only difference is that we replace $\mathscr{D}(\Omega)$ by $\mathscr{S}(\mathbb{R}^n)$.

Given an *operation* T on $\mathscr{S}(\mathbb{R}^n)$, assumed to be a linear map

$$T\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n),$$

that we would like to extend to tempered distributions.

Assume $S\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a linear and $\mathscr{S}$ continuous map, and that we have the *adjoint identity*:

$$\int_{\mathbb{R}^n} T(\phi)\psi \,\mathrm{d}x = \int_{\mathbb{R}^n} \phi S(\psi) \,\mathrm{d}x$$

holds for all ϕ, $\psi \in \mathscr{S}(\mathbb{R}^n)$.

The adjoint identity scheme in the tempered context

We can then define $\overline{T}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ for each $u \in \mathscr{S}'(\mathbb{R}^n)$ by the rule

$$\langle \overline{T}(u), \phi \rangle := \langle u, S(\phi) \rangle,\ \phi \in \mathscr{S}(\mathbb{R}^n).$$

We record that hereby $\overline{T}(u)\colon \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ is linear and $\mathscr{S}$ continuous, that is, $\overline{T}(u) \in \mathscr{S}'(\mathbb{R}^n)$, so $\overline{T}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is well-defined. By inspection we see that it is linear and $\mathscr{S}'$ *continuous*: if $u_j \to u$ in $\mathscr{S}'(\mathbb{R}^n)$, then also $\overline{T}(u_j) \to \overline{T}(u)$ in $\mathscr{S}'(\mathbb{R}^n)$.

Note that the adjoint identity ensures that the extension is consistent, $\overline{T}|_{\mathscr{S}(\mathbb{R}^n)} = T$ and so as in $\mathscr{D}$ context we shall in the sequel write T also for the extension $\overline{T}$.

The Fourier transform on tempered distributions

We have seen that the Fourier transform acts a linear and $\mathscr{S}$ continuous map $\mathcal{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. The product rule is therefore an adjoint identity and so we can define the Fourier transform on $\mathscr{S}'$ by the adjoint identity scheme: for $u \in \mathscr{S}'(\mathbb{R}^n)$ we define $\mathcal{F}u = \widehat{u}$ by the rule

$$\langle \widehat{u}, \phi \rangle := \langle u, \widehat{\phi} \rangle, \, \phi \in \mathscr{S}(\mathbb{R}^n).$$

Hereby $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is linear and $\mathscr{S}'$ continuous.

The adjoint identity ensures that our definition is consistent on Schwartz test functions, but what about our definition on $\mathrm{L}^1(\mathbb{R}^n)$, do we also have consistency there? – Let $f \in \mathrm{L}^1(\mathbb{R}^n)$ and let us compare our two definitions:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\mathrm{i}\xi \cdot x} \, \mathrm{d}x \; \text{ and } \; \langle \widehat{T_f}, \phi \rangle = \int_{\mathbb{R}^n} f \widehat{\phi} \, \mathrm{d}x, \, \phi \in \mathscr{S}(\mathbb{R}^n).$$

The product rule in L^1 ensures that they are the same: $T_{\widehat{f}} = \widehat{T_f}$.

The Fourier transform on tempered distributions

Example Find the Fourier transform of δ_a, where $a \in \mathbb{R}^n$.

For $\phi \in \mathscr{S}(\mathbb{R}^n)$ we have

$$\begin{aligned} \langle \widehat{\delta_a}, \phi \rangle = \langle \delta_a, \widehat{\phi} \rangle &= \widehat{\phi}(a) \\ &= \int_{\mathbb{R}^n} \phi(x) \mathrm{e}^{-\mathrm{i}a \cdot x} \, \mathrm{d}x, \end{aligned}$$

so

$$\widehat{\delta_a}(\xi) = \mathrm{e}^{-\mathrm{i}a \cdot \xi}.$$

In particular record the result for $a = 0$: $\widehat{\delta_0} = 1$.

Exercise Check that our definition of the Fourier transform on $\mathscr{S}'$ is consistent with the definition we gave for the Fourier transform of finite Borel measures in Lecture 1:

$$\widehat{T_\mu} = T_{\widehat{\mu}}$$

holds for all finite Borel measures μ on $\mathbb{R}^n$.

Extending other operations to tempered distributions

Because $\mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$ we can of course define many of the operations introduced in B4.3 also for tempered distributions. What is needed for the operation to produce a tempered distribution again is that the operation on $\mathscr{D}(\mathbb{R}^n)$ extends to a linear and $\mathscr{S}$ continuous map of $\mathscr{S}(\mathbb{R}^n)$ to itself. That is, we should have an adjoint identity in the $\mathscr{S}$ context.

This is easily seen to be the case with differentiation, where we define for a direction $1 \leq j \leq n$ and $u \in \mathscr{S}'(\mathbb{R}^n)$ the tempered distribution partial derivative $\partial_j u$ by the rule

$$\langle \partial_j u, \phi \rangle := -\langle u, \partial_j \phi \rangle,\ \phi \in \mathscr{S}(\mathbb{R}^n).$$

With this definition we can then, for each $u \in \mathscr{S}'(\mathbb{R}^n)$, make sense of $\partial^\alpha u$ and of $p(\partial)u$ as tempered distributions for any multi-index $\alpha \in \mathbb{N}_0^n$ and any differential operator $p(\partial)$.

Extending other operations to tempered distributions

Likewise, we can define the operations

- $\theta_* u$ for $\theta \in \mathrm{O}(n)$ (and in particular $\widetilde{u}$),
- dilations $d_r u$ and u_r for a scale factor $r > 0$,
- translation $\tau_h u$ for a vector $h \in \mathbb{R}^n$

on tempered distributions in a straight forward manner.

Example Let $u \in \mathscr{S}'(\mathbb{R})$. Then

$$\frac{\tau_h u - u}{h} \to u' \text{ in } \mathscr{S}'(\mathbb{R}^n) \text{ as } h \to 0.$$

However, some care is needed for *multiplication with* C^∞ *function*, where the multiplying function must be restricted. We pick up on this in the next lecture.