B4.4 Fourier Analysis HT21

Lecture 16: The Hilbert transform revisited

We follow up on examples from lectures 7 and 8 about the Hilbert transform. The material should be covered in Week 8.

The Hilbert transform was defined for each $\phi \in \mathscr{S}(\mathbb{R})$ in lecture 7 as

$$\mathcal{H}(\phi) := \frac{1}{\pi}\Big(\mathrm{pv}\big(\tfrac{1}{y}\big) * \phi\Big)(x) = \lim_{\varepsilon \searrow 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\phi(x-y)}{\pi y}\,\mathrm{d}y.$$

Hereby $\mathcal{H}\colon \mathscr{S}(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$ is linear and it is the most basic example of a *singular integral operator*. The distribution

$$\frac{1}{\pi}\mathrm{pv}\big(\frac{1}{x}\big)$$

is tempered and of order 1. Its Fourier transform is $-\mathrm{i}\,\mathrm{sgn}(\xi)$ and so we can use the extended convolution rule to define the Hilbert transform of a tempered distribution u whose Fourier transform $\widehat{u}$ is a moderate C^∞ function:

$$\mathcal{H}(u) = \mathcal{F}^{-1}_{\xi \to x}\Big(-\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{u}(\xi)\Big). \tag{1}$$

In fact, we can use this definition for all $u \in \mathscr{S}'(\mathbb{R})$ for which $-\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{u}(\xi)$ is a well-defined tempered distribution. But the question of its natural and maximal *domain* is subtle.

The Hilbert transform

An example where we can use (1) to define $\mathcal{H}(u)$ is when $u \in \mathrm{L}^1(\mathbb{R})$ since then Riemann-Lebesgue ensures $\widehat{u}$ is continuous. However, its Hilbert transform will not be integrable in general. In fact, in lecture 7 we saw examples where the Hilbert transform of Schwartz test functions are not integrable (we used the Riemann-Lebesgue lemma).

In this connection we also record:

Example 1 For any a, $b \in \mathbb{R}$ with $a < b$ we calculate

$$\mathcal{H}\big(\mathbf{1}_{(a,b)}\big)(x) = \frac{1}{\pi} \log \left| \frac{x-a}{x-b} \right|$$

and this also is not integrable on $\mathbb{R}$. Note that it is not bounded either. But you can check that it is in $\mathrm{L}^p(\mathbb{R})$ for each $p \in (1, \infty)$.

You might recall why this is not surprising when $p = 2$.

The Hilbert transform on L^2

Using Plancherel's theorem we proved in lecture 8 that $\mathcal{H}$ extends by continuity to $\mathsf{L}^2(\mathbb{R})$ and that hereby the extended map

$$\mathcal{H}\colon\ \mathsf{L}^2(\mathbb{R}) \to \mathsf{L}^2(\mathbb{R})$$

is unitary (isometric and onto). We can use (1) as definition again because $\widehat{u} \in \mathsf{L}^2(\mathbb{R})$ and so

$$-\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{u}(\xi) \in \mathsf{L}^2(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R}).$$

Because $\mathbf{1}_{(a,b)} \in \mathsf{L}^2(\mathbb{R})$ we therefore confirm our calculation that $\mathcal{H}(\mathbf{1}_{(a,b)}) \in \mathsf{L}^2(\mathbb{R})$.

The Hilbert transform on $\mathsf{L}^2(\mathbb{R})$ satisfies $\mathcal{H}^2 = -\mathrm{I}$, that is, minus the identity on $\mathsf{L}^2(\mathbb{R})$.

Proof. We use that for $\phi \in \mathscr{S}(\mathbb{R})$,

$$\widehat{\mathcal{H}(\phi)} = -\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{\phi}(\xi), \tag{2}$$

and since both the Hilbert and Fourier transforms are continuous on $\mathsf{L}^2(\mathbb{R})$, density of $\mathscr{S}(\mathbb{R})$ in $\mathsf{L}^2(\mathbb{R})$ allows us to extend (2) to $\phi \in \mathsf{L}^2(\mathbb{R})$. But then we get for $\phi \in \mathsf{L}^2(\mathbb{R})$ that

$$\begin{aligned}
\widehat{\mathcal{H}^2(\phi)}(\xi) &= -\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{\mathcal{H}(\phi)}(\xi) \\
&= -\mathrm{i}\,\mathrm{sgn}(\xi)\Big(-\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{\phi}(\xi)\Big) \\
&= -\widehat{\phi}(\xi)
\end{aligned}$$

concluding the proof. □

The Hilbert transform on L^p [Not examinable]

It can be shown that for each $p \in (1, \infty)$ there exists a constant $c_p > 0$ such that

$$\left\|\mathcal{H}(\phi)\right\|_p \leq c_p \left\|\phi\right\|_p \tag{3}$$

holds for all $\phi \in \mathscr{S}(\mathbb{R})$. We can therefore extend $\mathcal{H}$ to $\mathrm{L}^p(\mathbb{R})$ by continuity (recall the abstract extension theorem from lecture 8). Note that Example 1 shows that (3) cannot hold for $p = 1$ nor for $p = \infty$.

Can we use the formula (1) to calculate $\mathcal{H}(\phi)$ when $\phi \in \mathrm{L}^p(\mathbb{R})$?

The Hilbert transform on L^p [Not examinable]

We can use the formula (1) as definition of $\mathcal{H}(\phi)$ when $\phi \in \mathsf{L}^p(\mathbb{R})$ and $p \in [1,2]$. This is so because in these cases $\widehat{\phi}$ is a regular distribution and so we can make sense of

$$-\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{\phi}(\xi)$$

as a tempered distribution. We have already mentioned this for $p = 1$ and for $p = 2$. In the remaining cases $p \in (1,2)$ we have by Hausdorff-Young (that we quoted but didn't prove) that $\widehat{\phi} \in \mathsf{L}^q(\mathbb{R})$, where q is the Hölder conjugate exponent $q = p/(p-1)$. Thus

$$-\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{\phi}(\xi) \in \mathsf{L}^q(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R}).$$

However, when $p > 2$ the Fourier transform $\widehat{\phi} \in \mathscr{S}'(\mathbb{R})$ of $\phi \in \mathsf{L}^p(\mathbb{R})$ can be a distribution of higher order making it impossible to directly use (1) as definition of $\mathcal{H}(\phi)$. But obviously the abstract extension theorem and (3) still allow us to define the Hilbert transform in this situation – we just cannot rely on the formula (1).

A connection to holomorphic functions in the upper half-plane

For this we rely on the formula

$$\frac{1}{x+\mathrm{i}0} = -\pi \mathrm{i}\delta_0 + \mathrm{pv}\big(\tfrac{1}{x}\big). \tag{4}$$

Proof. Recall that we calculated the Fourier transform of Heaviside's function in lecture 6, example 2:

$$\widehat{H} = -\mathrm{i}\,\mathrm{pv}\big(\tfrac{1}{x}\big) + \pi\delta_0.$$

We will now calculate it in a different manner: put $H_\varepsilon(t) = \mathrm{e}^{-\varepsilon t}H(t)$ for $\varepsilon > 0$. Then $H_\varepsilon' = -\varepsilon H_\varepsilon + \delta_0$ in $\mathscr{S}'(\mathbb{R})$, so using the differentiation rule we get by Fourier transformation:

$$\widehat{H_\varepsilon}(x) = \frac{1}{\varepsilon + \mathrm{i}x} = \frac{-\mathrm{i}}{x - \mathrm{i}\varepsilon}.$$

Because $H_\varepsilon \to H$ in $\mathscr{S}'(\mathbb{R})$ as $\varepsilon \searrow 0$ we get by $\mathscr{S}'$ continuity of the Fourier transform that

$$\frac{-\mathrm{i}}{x-\mathrm{i}0} = -\mathrm{i}\,\mathrm{pv}\big(\tfrac{1}{x}\big) + \pi\delta_0.$$

A connection to holomorphic functions in the upper half-plane

To arrive at the formula (4) we apply the reflection in origin operation $\widetilde{(\cdot)}$ on the previous identity. $\square$

Now let $\phi \in \mathscr{S}(\mathbb{R})$ be *real-valued*. Define

$$\Phi(z) = \frac{\mathrm{i}}{\pi} \int_{-\infty}^{\infty} \frac{\phi(t)}{x - t + \mathrm{i}y} \, \mathrm{d}t$$

for $z = x + \mathrm{i}y \in \mathbb{H}$, where $\mathbb{H}$ is the open upper half-plane in $\mathbb{C}$. It is not difficult to check that $\Phi \colon \mathbb{H} \to \mathbb{C}$ is holomorphic and that we can rewrite it as

$$\Phi(z) = \frac{\mathrm{i}}{\pi} \left\langle \frac{1}{t + \mathrm{i}y}, \phi\big(x - \cdot\big) \right\rangle \tag{5}$$

Consider its real and imaginary parts. They clearly are a pair of conjugate harmonic functions on $\mathbb{H}$.

A connection to holomorphic functions in the upper half-plane

We have

$$\mathrm{Re}\big(\Phi(z)\big) = \big(P_y * \phi\big)(x) \text{ and } \mathrm{Im}\big(\Phi(z)\big) = \big(Q_y * \phi\big)(x)$$

where P_y is the *Poisson kernel* obtained by an L^1 dilation of

$$P(x) = \frac{1}{\pi\big(1 + x^2\big)}$$

and Q_y is the *conjugate Poisson kernel* obtained by an L^1 dilation of

$$Q(x) = \frac{x}{\pi\big(1 + x^2\big)}.$$

Note that $P(x) \geq 0$ and $\int_{\mathbb{R}} P(x)\,\mathrm{d}x = 1$, so $\big(P_y\big)_{y>0}$ is an approximate identity and we have $P_y * \phi \to \phi$ uniformly on $\mathbb{R}$ as $y \searrow 0$. What is the limit of $Q_y * \phi$? Complication: Q is *not* integrable on $\mathbb{R}$.

A connection to holomorphic functions in the upper half-plane

To find the limit as $y \searrow 0$ we return to (5) and (4):

$$
\begin{aligned}
\Phi(z) = \big(P_y * \phi\big)(x) + \mathrm{i}\big(Q_y * \phi\big)(x) \quad &= \quad \frac{\mathrm{i}}{\pi}\left\langle \frac{1}{t+\mathrm{i}y}, \phi(x-\cdot) \right\rangle \\
&\to \quad \frac{\mathrm{i}}{\pi}\left\langle -\pi\mathrm{i}\delta_0 + \mathrm{pv}\big(\tfrac{1}{t}\big), \phi(x-\cdot) \right\rangle \\
&= \quad \phi(x) + \mathrm{i}\mathcal{H}(\phi)(x)
\end{aligned}
$$

pointwise in $x \in \mathbb{R}$.