Fourier Analysis and PDEs HT20

Problem Sheet 1

Problem 1. Consider the following four functions on $\mathbb{R}$:

$$f_1(x) = \mathrm{e}^{-x^2+2x}, \quad f_2(x) = \mathrm{e}^{-x} H(x), \quad f_3(x) = \mathrm{e}^{-|x|}, \quad f_4(x) = \tfrac{1}{x^2+1},$$

where H is Heaviside's function.
(i) Verify that these functions all belong to $\mathrm{L}^1(\mathbb{R})$. Which of them belong to $\mathscr{S}(\mathbb{R})$ and which to $\mathrm{L}^2(\mathbb{R})$?
(ii) Calculate the Fourier transforms of these functions. Deduce *Laplace's integral*

$$\int_0^\infty \frac{\cos(x\xi)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{-|\xi|} \quad (\xi \in \mathbb{R}).$$

(iii) For each of the Fourier transforms $\hat{f}_j$, determine whether it is a function in $\mathscr{S}(\mathbb{R})$, in $\mathrm{L}^1(\mathbb{R})$, or in $\mathrm{L}^2(\mathbb{R})$.

Problem 2. Let $f \in \mathrm{L}^1(\mathbb{R}^n)$ and denote by $(\mathrm{e}_j)_{j=1}^n$ the standard basis for $\mathbb{R}^n$. For $\xi \in \mathbb{R}^n$ we write $\xi = \xi_1\mathrm{e}_1 + \cdots + \xi_n\mathrm{e}_n$. Show that if $\xi_j \neq 0$, then

$$\hat{f}(\xi) = -\int_{\mathbb{R}^n} f(x + \frac{\pi}{\xi_j}\mathrm{e}_j)\mathrm{e}^{-ix\cdot\xi}\,\mathrm{d}x,$$

and conclude that

$$|\hat{f}(\xi)| \leq \frac{1}{2}\int_{\mathbb{R}^n} |f(x) - f(x + \frac{\pi}{\xi_j}\mathrm{e}_j)|\,\mathrm{d}x.$$

Using that $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathrm{L}^1(\mathbb{R}^n)$ deduce the *Riemann-Lebesgue Lemma*: $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Problem 3. Let $t > 0$ and put $G_t(x) = \mathrm{e}^{-t|x|^2}$ for $x \in \mathbb{R}^n$. Use the Fourier transform to find a formula for the convolution $G_s * G_t$ for all $s, t > 0$.

Problem 4. Let $a > 0$ and $b, c \in \mathbb{R}$. Put $g(x) = \mathrm{e}^{-ax^2+bx+c}$, $x \in \mathbb{R}$. Calculate $\hat{g}$.

Problem 5. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Prove one of the following assertions and then derive the other:

(i) If $\phi(0) = 0$, then we may write $\phi = \sum_{j=1}^n x_j\phi_j$ with $\phi_j \in \mathscr{S}(\mathbb{R}^n)$.

(ii) If $\int_{\mathbb{R}^n}\phi\,\mathrm{d}x = 0$, then we may write $\phi = \sum_{j=1}^n \partial_j\phi_j$ with $\phi_j \in \mathscr{S}(\mathbb{R}^n)$.

It is ok if the students only do the one-dimensional case.

Problem 6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $|f(x)| \leq \mathrm{e}^{-|x|}$ for almost all $x \in \mathbb{R}$. Prove that the Fourier transform $\hat{f}$ cannot have compact support unless $f(x) = 0$ for almost all $x \in \mathbb{R}$. (*Hint: Use a Differentiation Rule to see that $\hat{f}$ is C^∞ and consider a suitable Taylor expansion.*)

Problem 7. (Optional) Let $f(x) = \mathrm{e}^{-|x|}$, $x \in \mathbb{R}^n$.
(a) Compute the Fourier transform $\hat{f}(\xi)$ when $n = 1$. Deduce for $\lambda \geq 0$ the identity

$$\mathrm{e}^{-\lambda} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+|\xi|^2} \mathrm{e}^{i\lambda\xi} \, \mathrm{d}\xi.$$

(b) Using $\frac{1}{1+|\xi|^2} = \int_0^\infty \mathrm{e}^{-(1+|\xi|^2)t} \, \mathrm{d}t$ and (a) show that for each $\lambda \geq 0$ the identity

$$\mathrm{e}^{-\lambda} = \int_0^\infty \frac{1}{\sqrt{\pi t}} \mathrm{e}^{-t-\frac{\lambda^2}{4t}} \, \mathrm{d}t$$

holds.
(c) Compute the Fourier transform $\hat{f}(\xi)$ in the general n-dimensional case, for instance by use of the formula from (b) with $\lambda = |x|$.

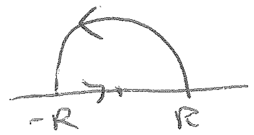
B4.4 Solutions to sheet 1 HT2020

(1) <u>(i)</u> Clear that $f_j \in L^1$: each f_j measurable, $f_j \geq 0$ and $\int f_j < \infty$. Likewise clear that $f_j \in L^2$.
Finally, $f_1 \in \mathcal{S}$ since it is C^∞ and for $k, \ell \in \mathbb{N}_0$ we have $x^j f_1^{(k)}(x) \to 0$ as $|x| \to \infty$.
$f_2, f_3 \notin \mathcal{S}$ since they're not diff. at 0,
$f_4 \notin \mathcal{S}$ since it's not rapidly decreasing, for instance $x^2 f_4(x) \to 1 \neq 0$ as $|x| \to \infty$.

<u>(ii)</u> $\hat{f}_1(\xi) = \sqrt{\pi}\, e^{1 - i\xi - \frac{1}{4}\xi^2}$, $\hat{f}_2(\xi) = \frac{2}{1+\xi^2}$,

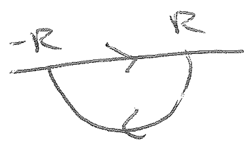
$\hat{f}_3(\xi) = \frac{2}{1+\xi^2}$ by straight forward calculations.

$\hat{f}_4(\xi)$ can be done by contour integration using:



if $\xi \leq 0$

$$\frac{e^{-i\xi z}}{1+z^2} \quad \&$$



if $\xi \geq 0$

Hereby $\hat{f}_4(\xi) = \pi e^{-|\xi|}$.

Since $\frac{\sin(x\xi)}{1+x^2}$ is odd, $\frac{\cos(x\xi)}{1+x^2}$ is even we deduce

$$\pi e^{-|\xi|} = 2\int_0^\infty \frac{\cos(x\xi)}{1+x^2}\,dx.$$

<u>(iii)</u> All $\hat{f}_j \in L^2$ while only $\hat{f}_1 \in \mathcal{S}$ and only $\hat{f}_1, \hat{f}_3, \hat{f}_4 \in L^1$.

$\varphi\rangle = (-1)^{|\alpha|} \langle u, D^\alpha\varphi\rangle = 0$

y $D^\alpha\varphi\big|_{\Omega\setminus \mathrm{supp}(u)} \in \mathcal{D}(\Omega\setminus \mathrm{supp}(u))$.

must have too

$$, \varphi\rangle = \sum_{|\alpha|\leq k} c_\alpha \langle D^\alpha u, \varphi\rangle = 0$$

n φ, showing that $p(D)u\big|_{\Omega\setminus\mathrm{supp}(u)} = 0$

ore $\mathrm{supp}(p(D)u) \subseteq \mathrm{supp}(u)$.

Look up Peetre's characterization of PDO.

$H:\mathbb{R}\to\mathbb{R}$ be the Heaviside

en $H \in \mathcal{D}'(\mathbb{R})$ and $\mathrm{supp}(H) = [0,\infty)$

has support $\{0\}$.

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(2) $x = (x_j, x') \in \mathbb{R}\times\mathbb{R}^{n-1}$ and note tha

bini yields: $\hat{f}(\xi) = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} f(x) e^{-ix_j\xi_j} dx_j e^{-ix'}$

$f(x + \frac{\pi}{\xi_j} e_j) e^{-i(x_j + \frac{\pi}{\xi_j})\xi_j} dx_j \, e^{-ix'\cdot\xi'} dx' =$

$e^{-i\pi} \int f(x + \frac{\pi}{\xi_j} e_j) e^{-ix\cdot\xi} dx$

$$= -\int_{\mathbb{R}^n} f(x+\tfrac{\pi}{\xi_j}e_j)\, e^{-ix\cdot\xi}\,dx.$$

Consequently, $\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^n}\left(f(x) - f(x+\frac{\pi}{\xi_j}e_j)\right)e^{-ix\cdot\xi}\,dx$

and so

$$|\hat{f}(\xi)| \leqq \frac{1}{2}\int_{\mathbb{R}^n} |f(x) - f(x+\tfrac{\pi}{\xi_j}e_j)|\,dx.$$

Let $\varepsilon>0$. Because $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ we can find $g\in\mathcal{D}(\mathbb{R}^n)$ so $\|f-g\|_1 < \frac{\varepsilon}{2}$.

Then $|\hat{f}(\xi)| \leqq |\hat{f}(\xi) - \hat{g}(\xi)| + |\hat{g}(\xi)| =$

$|(f-g)^\wedge(\xi)| + |\hat{g}(\xi)| \leqq \|f-g\|_1 + |\hat{g}(\xi)| < \frac{\varepsilon}{2} + |\hat{g}(\xi)|$.

Take $R>0$ so $\mathrm{spt}(g)\subset[-R,R]^n$. Since g is uniformly continuous we can find $\delta\in(0,1)$ so

$$|g(x)-g(y)| < \frac{\varepsilon}{(2(R+1))^n}$$

for $x,y\in\mathbb{R}^n$ with $|x-y|<\delta$.

Assume $\|\xi\|_\infty > \frac{\pi}{\delta}$. Then $\frac{\pi}{\|\xi\|_\infty} < \delta$, hence for $j\in\{1,\dots,n\}$ with $|\xi_j| = \|\xi\|_\infty$ we get by the above

$$|\hat{g}(\xi)| \leqq \frac{1}{2}\int_{\mathbb{R}^n} |g(x) - g(x+\tfrac{\pi}{\xi_j}e_j)|\,dx$$

$$= \frac{1}{2}\int_{(-R-1,R+1)^n} |g(x) - g(x+\tfrac{\pi}{\xi_j}e_j)|\,dx$$

$$\leq \frac{1}{2}\left(2(R+1)\right)^n \frac{\varepsilon}{(2(R+1))^n} = \frac{\varepsilon}{2}.$$

Consequently we get for all $\xi \in \mathbb{R}^n$ with $\|\xi\|_\infty > \frac{\pi}{\delta}$ that $|\hat{f}(\xi)| < \varepsilon$. $\square$

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(3) From Auxiliary Lemma for Fourier Inversion Formula in $\mathcal{S}$ we get

$$\hat{G}_t(\xi) = \left(\frac{\pi}{t}\right)^{\frac{n}{2}} e^{-\frac{|\xi|^2}{4t}},$$

so by a Convolution Rule,

$$\widehat{G_s * G_t}(\xi) = \hat{G}_s(\xi)\,\hat{G}_t(\xi) = \left(\frac{\pi^2}{st}\right)^{\frac{n}{2}} e^{-\left(\frac{1}{4s}+\frac{1}{4t}\right)|\xi|^2}$$

Hence, applying the Auxilary Lemma once more,

$$\widehat{\widehat{G_s * G_t}}(\xi) = \mathcal{F}_{\xi\to x}\left\{\left(\frac{\pi^2}{st}\right)^{\frac{n}{2}} e^{-\left(\frac{1}{4s}+\frac{1}{4t}\right)|\xi|^2}\right\}$$

$$= \left(\frac{\pi^2}{st}\right)^{\frac{n}{2}} \left(\frac{\pi}{\frac{1}{4s}+\frac{1}{4t}}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4\left(\frac{1}{4s}+\frac{1}{4t}\right)}}$$

$$= (2\pi)^n \left(\frac{\pi}{s+t}\right)^{\frac{n}{2}} e^{-\frac{st}{s+t}|x|^2}.$$

Finally from the Fourier Inversion Formula for $\mathcal{S}$, $\widehat{\widehat{(\cdot)}} = (2\pi)^n \widetilde{(\cdot)}$, so

$$G_s * G_t(x) = \left(\frac{\pi}{s+t}\right)^{\frac{n}{2}} e^{-\frac{st}{s+t}|x|^2}$$

<u>Note</u> If $H_t := (2\pi)^{-n}\hat{G}_t$, then we have the <u>semigroup property</u>: $H_s * H_t = H_{s+t}$, $\forall\, s,t>0$.

4. Write $-ax^2+bx+c = -a\left(x-\frac{b}{2a}\right)^2+\frac{b^2}{4a}+c$ 4/13

and use formula for Fourier transform of Gaussian (= Auxiliary Lemma):

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{-a\left(x-\frac{b}{2a}\right)^2+\frac{b^2}{4a}+c-ix\xi}\,dx =$$

$$\int_{-\infty}^{\infty} e^{-a\left(x-\frac{b}{2a}\right)^2-ix\xi}\,dx\; e^{\frac{b^2}{4a}+c} \overset{x\to x-\frac{b}{2a}}{=}$$

$$\int_{-\infty}^{\infty} e^{-ax^2-i\left(x+\frac{b}{2a}\right)\xi}\,dx\; e^{\frac{b^2}{4a}+c} =$$

$$\int_{-\infty}^{\infty} e^{-ax^2-ix\xi}\,dx\; e^{-i\frac{b}{2a}\xi+\frac{b^2}{4a}+c} =$$

$$\left(\frac{\pi}{a}\right)^{\frac{1}{2}} e^{-\frac{|\xi|^2}{4a}-i\frac{b}{2a}\xi+\frac{b^2}{4a}+c}.$$

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5. (a) ~~BW~~ - see LN Theorem 11.12.

~~(b) Clearly $\delta_x \in \mathcal{S}'(\mathbb{R}^n)$ and for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,~~

~~$\langle\hat{\delta}_x, \varphi\rangle = \langle\delta_x, \hat{\varphi}\rangle = \hat{\varphi}(x) =$~~

~~$\int_{\mathbb{R}^n}\varphi(y)e^{-ix\cdot y}\,dy = \langle e^{-ix\cdot(\cdot)}, \varphi\rangle,$~~

~~so $\hat{\delta}_x = e^{-ix\cdot(\cdot)}$. From the Fourier~~

(5) We establish (i) and obtain (ii) by use of the Fourier transform.

The proof of (i) is by induction on the dimension $n \in \mathbb{N}$.

<u>Base case $n=1$.</u> Fix $\phi \in \mathcal{S}(\mathbb{R})$ with $\phi(0)=0$.

Then by FTC:
$$\phi(x) = \int_0^1 \frac{d}{dt}\phi(tx)\,dt = \int_0^1 \phi'(tx)\,dt\,x.$$

Let $\chi = \rho * \mathbb{1}_{(-2,2)}$, where ρ is the standard mollifier kernel on $\mathbb{R}$. Then $\chi \in \mathcal{D}(\mathbb{R})$ and $\chi = 1$ near $(-1,1)$. Put

$$a(x) = \int_0^1 \phi'(tx)\,dt\,\chi(x),$$
$$b(x) = \phi(x)\frac{1-\chi(x)}{x} \qquad (x \in \mathbb{R})$$

Then $a, b \in C^\infty(\mathbb{R})$, $\operatorname{supp}(a) \subseteq \operatorname{supp}(\chi)$ so $a \in \mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$. For b we let $k, l \in \mathbb{N}_0$ and have from Leibniz:

$$b^{(l)}(x) = \sum_{i=0}^{l} \binom{l}{i} \phi^{(i)}(x) \left(\frac{1-\chi(x)}{x}\right)^{(l-i)}.$$

Here $\left|\left(\frac{1-\chi}{x}\right)^{(l-i)}\right| \leq \overline{S}_{0,l}(\chi)$ so we may estimate

$$\bar{s}_{k,\ell}(b) \leq 2^{\ell}\, \bar{s}_{k,\ell}(\phi)\, \bar{s}_{0,\ell}(\chi) < \infty$$

so $b \in \mathcal{S}(\mathbb{R})$. It follows that $\phi_1 = a + b \in \mathcal{S}(\mathbb{R})$ and $\phi = \phi_1 x$.

Let $n \in \mathbb{N}$, $n > 1$, and suppose (i) holds in dimension $n-1$. Fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi(0) = 0$ and write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. For fixed $x' \in \mathbb{R}^{n-1}$, $\phi(x_1, x') - \phi(0, x') =$

$$\left(\int_0^1 \partial_1 \phi(tx_1, x')\, dt\, \chi(x_1) + (\phi(x_1, x') - \phi(0, x')) \frac{1 - \chi(x_1)}{x_1} \right) x_1$$

so

$$\phi(x) = \left(\int_0^1 \partial_1 \phi(tx_1, x')\, dt\, \chi(x_1) + \phi(x) \frac{1 - \chi(x_1)}{x_1} \right) x_1$$

$$+ \phi(0, x')\chi(x_1).$$

It follows as in 1-dimensional case that

$$\phi_1(x) = \int_0^1 \partial_1 \phi(tx_1, x')\, dt\, \chi(x_1) + \phi(x) \frac{1 - \chi(x_1)}{x_1}$$

belongs to $\mathcal{S}(\mathbb{R}^n)$. Since $\phi(0, \cdot) \in \mathcal{S}(\mathbb{R}^{n-1})$ and $\phi(0, 0') = 0$ we find $\phi_2', \dots, \phi_n' \in \mathcal{S}(\mathbb{R}^{n-1})$ by the induction hypothesis so

$$\phi(0,x') = \sum_{j=2}^{n} \phi_j'(x')x_j .$$

Consequently if $\phi_j(x) = \phi_j'(x')\chi(x_1)$, then $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ and

$$\phi = \sum_{j=1}^{n} \phi_j x_j \quad , \quad \text{as required.}$$

We derive (ii) from (i) by use of the Fourier transform: Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi = 0$. Then $\hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{\phi}(0) = \int_{\mathbb{R}^n} \phi = 0$, so for some $\psi_j \in \mathcal{S}(\mathbb{R}^n)$

$$\hat{\phi} = \sum_1^n \psi_j \xi_j .$$

By the (reverse) diff. rule

$$\hat{\hat{\phi}} = \sum_1^n \mathcal{F}_{\xi\to x}(\psi_j \xi_j) = \sum_1^n \partial_j (i\hat{\psi}_j),$$

where $i\hat{\psi}_j \in \mathcal{S}(\mathbb{R}^n)$. By the Fourier inversion formula on $\mathcal{S}(\mathbb{R}^n)$,

$$\tilde{\phi} = \frac{1}{(2\pi)^n}\hat{\hat{\phi}} = \sum_1^n \partial_j\left(\frac{i}{(2\pi)^n}\hat{\psi}_j\right) , \quad \text{or}$$

$$\phi = \sum_1^n \partial_j \phi_j \quad \text{with} \quad \phi_j = \frac{1}{(2\pi)^n i}\widetilde{\hat{\psi}_j} \in \mathcal{S}(\mathbb{R}^n).$$

Clearly $|x|^k f \in L^1(\mathbb{R})$ for all $k \in \mathbb{N}_0$ so $\hat{f} \in C^k(\mathbb{R})$ with $\hat{f}^{(k)}(\xi) = \int_{-\infty}^{\infty} f(x)(-ix)^k e^{-ix\xi}\,dx$

hence $$|\hat{f}^{(k)}(\xi)| \leq \int_{-\infty}^{\infty} |x|^k |f(x)|\,dx \leq 2\int_0^{\infty} x^k e^{-x}\,dx$$
$$= 2 \cdot k!.$$

If $\operatorname{spt}(\hat{f}) \neq \mathbb{R}$ then we may select $\xi_0 \in \partial(\operatorname{spt}(\hat{f}))$. Clearly $\hat{f}^{(k)}(\xi_0) = 0$ for all k, so by Taylor's formula

$$\hat{f}(\xi) = \sum_{k=0}^{n} \frac{\hat{f}^{(k)}(\xi_0)}{k!}(\xi - \xi_0)^k + R_n(\xi, \xi_0)$$

$$= R_n(\xi, \xi_0)$$

where $$R_n(\xi, \xi_0) = \frac{\hat{f}^{(n+1)}(\xi_0 + \theta(\xi - \xi_0))}{(n+1)!}(\xi - \xi_0)^{n+1}$$

for some $\theta = \theta_n(\xi, \xi_0) \in (0,1)$. Now we may estimate

$$|R_n(\xi, \xi_0)| \leq \frac{2 \cdot (n+1)!}{(n+1)!}|\xi - \xi_0|^{n+1} = 2|\xi - \xi_0|^{n+1}.$$

and so if $|\xi - \xi_0| < 1$, then

$$R_n(\xi, \xi_0) \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, $\hat{f}(\xi) = 0$ for all $\xi \in (\xi_0 - 1, \xi_0 + 1)$.

But then $\xi_0 \notin \partial(\operatorname{spt}(\hat{f}))$ and we conclude that either $\operatorname{spt}(\hat{f}) = \emptyset$ or $\operatorname{spt}(\hat{f}) = \mathbb{R}$.

~~Distribution Theory and Fourier Analysis Sheet 4 solutions~~

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(a) Clearly $f \in L^1(\mathbb{R}^n)$ (any $n \in \mathbb{N}$) so

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix.\xi} \, dx = \int_{\mathbb{R}^n} e^{-|x| - ix.\xi} \, dx.$$

i) $\underline{n=1}$: $\hat{f}(\xi) = \int_{-\infty}^{0} e^{(1-i\xi)x} \, dx + \int_{0}^{\infty} e^{-(1+i\xi)x} \, dx \overset{FTC}{=}$

$\frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+|\xi|^2}$, and so by Fourier Inversion Formula in $\mathcal{S}'(\mathbb{R})$ (note $f \notin \mathcal{S}$)

$$e^{-|x|} = \frac{1}{2\pi} \tilde{\mathcal{F}}(\hat{f})(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+|\xi|^2} e^{ix\xi} \, d\xi.$$

Take $\lambda = |x|$ yields required identity.

(b) Now for $\lambda \geqslant 0$:

$$e^{-\lambda} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+|\xi|^2} e^{i\lambda\xi} \, d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-(1+|\xi|^2)t} \, dt \, e^{i\lambda\xi} \, d\xi$$

$$\overset{\text{Fubini}}{=} \frac{1}{\pi} \int_{0}^{\infty} \underbrace{\int_{-\infty}^{\infty} e^{-t|\xi|^2} e^{i\lambda\xi} \, d\xi}_{\mathcal{F}(e^{-t|\xi|^2})(-\lambda)} \, e^{-t} \, dt$$

$G_t = e^{-t|\cdot|^2}$

see proof for Fourier inversion in $\mathcal{S}$

$$= \frac{1}{\pi} \int_{0}^{\infty} \left(\frac{\pi}{t}\right)^{\frac{1}{2}} e^{-\frac{\lambda^2}{4t}} e^{-t} \, dt$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{\pi t}} e^{-t-\frac{\lambda^2}{4t}} \, dt, \text{ as required.}$$

(c) $\hat{f}(\xi) \overset{(b)}{=} \int_{\mathbb{R}^n} \int_{0}^{\infty} \frac{1}{\sqrt{\pi t}} e^{-t-\frac{|x|^2}{4t}} \, dt \, e^{-ix.\xi} \, dx$

$$\overset{\text{Fubini}}{=} \int_{0}^{\infty} \frac{1}{\sqrt{\pi t}} e^{-t} \hat{G}_{(4t)^{-1}}(\xi) \, dt =$$

$$\int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-t} \left(\frac{\pi}{\frac{1}{4t}}\right)^{\frac{n}{2}} e^{-4t\frac{|\xi|^2}{4}}\, dt =$$

$$s = (1+|\xi|^2)t$$

$$2^n \pi^{\frac{n-1}{2}} \int_0^\infty t^{\frac{n-1}{2}} e^{-t|\xi|^2 - t}\, dt =$$

$$2^n \pi^{\frac{n-1}{2}} \int_0^\infty \left(\frac{s}{1+|\xi|^2}\right)^{\frac{n-1}{2}} e^{-s} \frac{ds}{1+|\xi|^2} =$$

$$2^n \pi^{\frac{n-1}{2}} \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}} \int_0^\infty s^{\frac{n-1}{2}} e^{-s}\, ds =$$

$$2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}.$$

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(a) $g \in \mathcal{S}'(\mathbb{R}^n)$ by Lemma 12.2. Calculating in polar coordinates we see that

$$\int_{\mathbb{R}^n} |g(x)|\, dx = \omega_{n-1} \int_0^\infty (1+r^2)^{-\frac{\alpha}{2}} r^{n-1}\, dr < \infty$$

iff $-2\frac{\alpha}{2} + n - 1 < -1$, so iff $\alpha > n$.

(b)
$$\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-(1+|x|^2)t}\, dt \overset{s=(1+|x|^2)t}{=}$$

$$\int_0^\infty \left(\frac{s}{1+|x|^2}\right)^{\frac{\alpha}{2}-1} e^{-s} \frac{ds}{1+|x|^2} = (1+|x|^2)^{-\frac{\alpha}{2}} \int_0^\infty s^{\frac{\alpha}{2}-1} e^{-s}\, ds$$

$$= \Gamma\left(\frac{\alpha}{2}\right) g(x), \text{ so } c = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \in (0, \infty).$$

(c) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then using (b) and Fubini we calculate: