(1) For $\varphi \in \mathscr{S}(\mathbb{R})$ and $\mathbb{1}_{(-t,t)} \in L^1(\mathbb{R})$ the Product Rule gives $\int_{\mathbb{R}} \hat{\varphi}\, \mathbb{1}_{(-t,t)}\, dx = \int_{\mathbb{R}} \varphi\, \widehat{\mathbb{1}_{(-t,t)}}\, dx$,

and so

$$(*) \quad \int_{-t}^{t} \hat{\varphi}\, dx = 2\int_{-\infty}^{\infty} \varphi(x) \frac{\sin(tx)}{x}\, dx$$

holds for all $t > 0$, $\varphi \in \mathscr{S}(\mathbb{R})$.

Since $\mathbb{1}_{(-t,t)} \xrightarrow[t\to\infty]{} \mathbb{1}_{\mathbb{R}}$ in $\mathscr{S}'(\mathbb{R})$ and $\mathcal{F}$ is $\mathscr{S}'$-continuous,

$$\langle \widehat{\mathbb{1}_{\mathbb{R}}}, \varphi \rangle = \lim_{t\to\infty} 2\int_{-\infty}^{\infty} \varphi(x) \frac{\sin(tx)}{x}\, dx$$

for each $\varphi \in \mathscr{S}(\mathbb{R})$. From $\hat{\delta}_0 = \mathbb{1}_{\mathbb{R}}$ and the Fourier Inversion Formula in $\mathscr{S}'$,

$$2\pi \tilde{\delta}_0 = \hat{\hat{\delta}}_0 = \widehat{\mathbb{1}_{\mathbb{R}}}, \quad \text{hence, as } \tilde{\delta}_0 = \delta_0,$$

$$\langle 2\pi \delta_0, \varphi \rangle = \lim_{t\to\infty} 2\int_{-\infty}^{\infty} \varphi(x) \frac{\sin(tx)}{x}\, dx$$

and the conclusion $\frac{\sin(tx)}{x} \xrightarrow[t\to\infty]{} \pi\delta_0$ in $\mathscr{S}'(\mathbb{R})$ follows.

② Let $\phi \in \mathcal{S}$. For the standard mollifier ρ on $\mathbb{R}^n$ put $\phi_j = \rho * \mathbb{1}_{B_j(0)}$ $(j \in \mathbb{N})$.

Then $\phi_j \in \mathcal{D}$ and for $\alpha, \beta \in \mathbb{N}_0^n$:

$$|x^\alpha \partial^\beta (\phi - \phi_j)| = |x^\alpha \partial^\beta [\phi (\rho * \mathbb{1}_{\mathbb{R}^n \setminus B_j(0)})]|$$

$$\overset{\text{Leibniz}}{=} |x^\alpha \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^\gamma \phi \, ((\partial^{\beta-\gamma} \rho) * \mathbb{1}_{\mathbb{R}^n \setminus B_j(0)})|$$

$$\leq \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} |x^\alpha \partial^\gamma \phi| \cdot |(\partial^{\beta-\gamma} \rho) * \mathbb{1}_{\mathbb{R}^n \setminus B_j(0)}|.$$

Note

$$|[(\partial^{\beta-\gamma} \rho) * \mathbb{1}_{\mathbb{R}^n \setminus B_j(0)}](x)| \leq \int_{|y| \geq j} |\partial^{\beta-\gamma} \rho|(x-y)\, dy$$

and this is 0 when $|x| \leq j-1$ since ρ is supported in $B_1(0)$. Thus

$$|[(\partial^{\beta-\gamma} \rho) * \mathbb{1}_{B_j(0)^c}](x)| \leq \mathbb{1}_{B_{j-1}(0)^c}(x) \int_{\mathbb{R}^n} |\partial^{\beta-\gamma} \rho|,$$

and

$$|x^\alpha \partial^\beta (\phi - \phi_j)| \leq \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} |x^\alpha \partial^\gamma \phi| \, \mathbb{1}_{B_{j-1}(0)^c}(x) \int_{\mathbb{R}^n} |\partial^{\beta-\gamma} \rho|$$

$$= \sum_{\gamma \leqq \beta} \binom{\beta}{\gamma} |x|^2 |x^\alpha \partial^\gamma \phi| \frac{\mathbb{1}_{B_{j-1}(0)^c}(x)}{|x|^2} \int_{\mathbb{R}^n} |\partial^{\beta-\gamma} \rho| .$$

$$\leqq \sum_{\gamma \leqq \beta} \binom{\beta}{\gamma} \sum_{j=1}^{n} |x^{\alpha+2e_j} \partial^\gamma \phi| \frac{1}{(j-1)^2} \int_{\mathbb{R}^n} |\partial^{\beta-\gamma} \rho|$$

$$\leqq \left\{ \sum_{\gamma \leqq \beta} \binom{\beta}{\gamma} \sum_{j=1}^{n} S_{\alpha+2e_j,\gamma}(\phi) \int_{\mathbb{R}^n} |\partial^{\beta-\gamma} \rho| \right\} \frac{1}{(j-1)^2}$$

$\longrightarrow 0$ uniformly in $x \in \mathbb{R}^n$,

so $\phi_j \to \phi$ in $\mathcal{S}$.

4/15

$$\int_0^\infty \frac{1}{\sqrt{\pi t}}\, e^{-t} \left(\frac{\pi}{\frac{1}{4t}}\right)^{\frac{n}{2}} e^{-4t\frac{|\xi|^2}{4}}\, dt \overset{s=(1+|\xi|^2)t}{=}$$

$$2^n \pi^{\frac{n-1}{2}} \int_0^\infty t^{\frac{n-1}{2}} e^{-\cancel{4}t|\xi|^2-t}\, dt =$$

$$2^n \pi^{\frac{n-1}{2}} \int_0^\infty \left(\frac{s}{1+|\xi|^2}\right)^{\frac{n-1}{2}} e^{-s} \frac{ds}{1+|\xi|^2} =$$

$$2^n \pi^{\frac{n-1}{2}} \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}} \int_0^\infty s^{\frac{n-1}{2}} e^{-s}\, ds =$$

$$2^n \pi^{\frac{n-1}{2}}\, \Gamma\!\left(\frac{n+1}{2}\right) \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}.$$

4

(a) $g \in \mathcal{S}'(\mathbb{R}^n)$ by Lemma 12.2. Calculating in polar coordinates we see that

$$\int_{\mathbb{R}^n} |g(x)|\, dx = \omega_{n-1} \int_0^\infty (1+r^2)^{-\frac{\alpha}{2}} r^{n-1}\, dr < \infty$$

iff $-2\frac{\alpha}{2} + n - 1 < -1$, so iff $\alpha > n$.

(b)
$$\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-(1+|x|^2)t}\, dt \overset{s=(1+|x|^2)t}{=}$$

$$\int_0^\infty \left(\frac{s}{1+|x|^2}\right)^{\frac{\alpha}{2}-1} e^{-s} \frac{ds}{1+|x|^2} = (1+|x|^2)^{-\frac{\alpha}{2}} \int_0^\infty s^{\frac{\alpha}{2}-1} e^{-s}\, ds$$

$$= \Gamma\!\left(\frac{\alpha}{2}\right) g(x), \quad \text{so} \quad c = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \in (0,\infty).$$

(c) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then using (b) and Fubini we calculate:

$$\langle \hat{g}, \varphi \rangle = \langle g, \hat{\varphi} \rangle = c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \int_{\mathbb{R}^n} e^{-t|x|^2} \hat{\varphi}(x)\,dx\,dt$$

Recall $G_t(x) = e^{-t|x|^2}$, $\hat{G}_t(x) = \left(\frac{\pi}{t}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ (see Theorem 10.3) so by the Product Rule:

$$\int_{\mathbb{R}^n} G_t \hat{\varphi}\,dx = \int_{\mathbb{R}^n} \hat{G}_t \varphi\,dx,$$

hence

$$\langle \hat{g}, \varphi \rangle = c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \int_{\mathbb{R}^n} \hat{G}_t(x)\varphi(x)\,dx\,dt$$

$$= \int_{\mathbb{R}^n} \left(c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \hat{G}_t(x)\,dt \right) \varphi(x)\,dx$$

by Fubini, since

$$c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \hat{G}_t(x)\,dt > 0 \quad \forall x$$

and

$$\int_{\mathbb{R}^n} c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \hat{G}_t(x)\,dt\,dx \overset{\text{Tonelli}}{=}$$

$$c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \int_{\mathbb{R}^n} \hat{G}_t(x)\,dx\,dt = c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} (2\pi)^n\,dt$$

See pf for Th. 10.3

$$= c\,\Gamma\!\left(\tfrac{\alpha}{2}\right)(2\pi)^n = (2\pi)^n < \infty.$$

Thus $\hat{g}(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \hat{G}_t(x)\,dt$,

a positive integrable function, as required.

(4)

(i) We use $\widehat{\mathbb{1}_{\mathbb{R}}} = 2\pi\delta_0$ and the translation rule $(ue^{-ixh})^\wedge = \tau_h \hat{u}$:

$$\widehat{\cos} = \frac{1}{2}(\widehat{e^{ix}} + \widehat{e^{-ix}}) = \frac{1}{2}(2\pi\tau_{-1}\delta_0 + 2\pi\tau_1\delta_0)$$
$$= \pi(\delta_{-1} + \delta_1).$$

$$\widehat{\sin} = \frac{1}{2i}(\widehat{e^{ix}} - \widehat{e^{-ix}}) = \frac{\pi}{i}(\delta_{-1} - \delta_1)$$
$$= \pi i(\delta_1 - \delta_{-1}).$$

$$\cos^2 x = \frac{1}{4}(e^{i2x} + e^{-i2x} + 2), \quad \text{so}$$
$$\widehat{\cos^2} = \frac{1}{4}2\pi(\delta_{-2} + \delta_2 + 2\delta_0) = \frac{\pi}{2}(\delta_{-2}+\delta_2+2\delta_0).$$

$$\cos^k x = 2^{-k}\sum_{s=0}^{k}\binom{k}{s} e^{isx - i(k-s)x} = 2^{-k}\sum_{s=0}^{k}\binom{k}{s}e^{i(2s-k)x},$$

so
$$\widehat{\cos^k} = 2^{-k}\sum_{s=0}^{k}\binom{k}{s}\tau_{k-2s}\delta_0$$
$$= 2^{-k}\sum_{s=0}^{k}\binom{k}{s}\delta_{k-2s}.$$

(ii) From EX 1.4: $\widehat{\mathbb{1}_{(-1,1)}} = 2\,\mathrm{sinc}$, so by the Fourier Inversion Formula in $\mathcal{S}'$:

$$\widehat{2\,\mathrm{sinc}} = \widehat{\widehat{\mathbb{1}_{(-1,1)}}} = 2\pi\,\widetilde{\mathbb{1}_{(-1,1)}} = 2\pi\,\mathbb{1}_{(-1,1)},$$

ie $\widehat{\mathrm{sinc}} = \pi\,\mathbb{1}_{(-1,1)}$.

(iii) See EX 1.61 in Lecture Notes:

$$\widehat{H} = -i\,\mathrm{pv}\left(\tfrac{1}{\xi}\right) + \pi\delta_0$$

(iv) $\widehat{x^+} = \widehat{xH} = i\widehat{H}'$ by differentiation rule, so $\widehat{x^+} = -\mathrm{fp}\left(\frac{1}{\xi^2}\right) + \pi i\,\delta_0'$.

Note $x^- = \widetilde{x^+}$ and $|x| = x^+ + x^- = x^+ + \widetilde{x^+}$,

so $\widehat{|x|} = \widehat{x^+} + \widehat{\widetilde{x^+}} = -2\,\mathrm{fp}\left(\frac{1}{\xi^2}\right)$,

since δ_0' is odd, $\mathrm{fp}\left(\frac{1}{\xi^2}\right)$ is even.

$$\sin|x| = \mathrm{sgn}(x)\sin x = \frac{1}{2i}\left(\mathrm{sgn}(x)\,e^{ix} - \mathrm{sgn}(x)\,e^{-ix}\right)$$

so by translation rule,

$$\widehat{\sin|x|} = \frac{1}{2i}\left(\tau_{-1}\widehat{\mathrm{sgn}} - \tau_1\widehat{\mathrm{sgn}}\right).$$

From EX 1.60 in Lecture Notes:

$$\widehat{\mathrm{pv}\left(\tfrac{1}{x}\right)} = -i\pi\,\mathrm{sgn}$$

so by Fourier Inversion Formula in $\mathcal{S}'$,

$$-i\pi\,\widehat{\mathrm{sgn}} = \widehat{\widehat{\mathrm{pv}\left(\tfrac{1}{x}\right)}} = 2\pi\,\widetilde{\mathrm{pv}\left(\tfrac{1}{\xi}\right)} = -2\pi\,\mathrm{pv}\left(\tfrac{1}{\xi}\right),$$

hence $\widehat{\mathrm{sgn}} = -2i\,\mathrm{pv}\left(\frac{1}{\xi}\right)$ and

$$\widehat{\sin|x|} = \mathrm{pv}\left(\frac{1}{\xi+1}\right) - \mathrm{pv}\left(\frac{1}{\xi-1}\right).$$

Finally we use Fourier Inversion Formula in $\mathcal{S}'$ on (iv): $\widehat{|x|} = -2\,\mathrm{fp}\left(\frac{1}{\xi^2}\right)$,

so $$-2\,\widehat{\mathrm{fp}\left(\frac{1}{x^2}\right)} = \widehat{\widehat{|x|}} = 2\pi\,\widetilde{|\xi|} = 2\pi|\xi|,$$

or $$\widehat{\mathrm{fp}\left(\frac{1}{x^2}\right)} = -\pi|\xi|.$$

(5) The integral is clearly convergent absolutely and we note that since the integrand is even

$$I = \int_0^\infty \frac{\sin(ax)\sin(bx)}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(ax)\sin(bx)}{x^2}\,dx.$$

Using sinc we express it as

$$I = \frac{ab}{2}\int_{-\infty}^{\infty} \operatorname{sinc}(ax)\operatorname{sinc}(bx)\,dx$$

and since sinc $\in L^2$ we may use Parseval's formula and the dilation rule

$$I = \frac{ab}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{d_a \operatorname{sinc}}(\xi)\,\overline{\widehat{d_b \operatorname{sinc}}(\xi)}\,d\xi$$

$$\overset{4(ii)}{=} \frac{ab}{4\pi}\int_{-\infty}^{\infty} (\pi \mathbb{1}_{(-1,1)})_a(\xi)\,(\pi \mathbb{1}_{(-1,1)})_b(\xi)\,d\xi$$

$$= \frac{ab}{4\pi}\int_{-\infty}^{\infty} \frac{\pi}{a}\mathbb{1}_{(-a,a)}(\xi)\,\frac{\pi}{b}\mathbb{1}_{(-b,b)}(\xi)\,d\xi$$

$$= \frac{\pi}{4}\int_{-\infty}^{\infty} \mathbb{1}_{(-a,a)}(\xi)\,\mathbb{1}_{(-b,b)}(\xi)\,d\xi = \frac{\pi}{4}\begin{cases} 2a & \text{if } a \le b \\ 2b & \text{if } a > b \end{cases}$$

$$= \frac{\pi}{2}\min\{a,b\}.$$

(6) Clearly $H^\varepsilon \in L^1 \subset \mathcal{S}'$ and from Leibniz

$$\frac{d}{dx} H^\varepsilon = \frac{d}{dx}\left(e^{-\varepsilon x} H\right) = -\varepsilon H^\varepsilon + \delta_0 \quad \text{in } \mathcal{S}'.$$

By differentiation rule: $i\xi \widehat{H^\varepsilon} = -\varepsilon \widehat{H^\varepsilon} + 1$,

so $\widehat{H^\varepsilon} = \dfrac{1}{i\xi + \varepsilon} = \dfrac{1}{i} \dfrac{1}{\xi - i\varepsilon}$.

Now $H^\varepsilon \xrightarrow[\varepsilon \searrow 0]{} H$ in $\mathcal{S}'$ and $\mathcal{F}$ is $\mathcal{S}'$ continuous, so $\widehat{H^\varepsilon} \xrightarrow[\varepsilon \searrow 0]{} \widehat{H}$ in $\mathcal{S}'$ and

so $\dfrac{1}{\xi - i\varepsilon} \xrightarrow[\varepsilon \searrow 0]{} i\widehat{H}$ in $\mathcal{S}'$.

Note that since $\widetilde{(\cdot)}$ is $\mathcal{S}'$ continuous also

$$\frac{1}{-\xi - i\varepsilon} = -\frac{1}{\xi + i\varepsilon} \xrightarrow[\varepsilon \searrow 0]{} i\widetilde{\widehat{H}} \quad \text{in } \mathcal{S}'.$$

Now, writing $(\xi \pm i0)^{-1} = \lim_{\varepsilon \searrow 0} \dfrac{1}{\xi \pm i\varepsilon}$,

$$\begin{aligned}(\xi + i0)^{-1} - (\xi - i0)^{-1} &= -i\widetilde{\widehat{H}} - i\widehat{H} \\ &= -i\,\widehat{(\widetilde{H} + H)} \\ &= -i\,\widehat{\mathbb{1}_{\mathbb{R}}}\end{aligned}$$

Since $\widehat{\delta_0} = \mathbb{1}_{\mathbb{R}}$ we get by the Fourier Inversion Formula in $\mathcal{S}'$, $\widehat{\mathbb{1}_{\mathbb{R}}} = 2\pi\delta_0$, and therefore $(\xi + i0)^{-1} - (\xi - i0)^{-1} = -2\pi i\,\delta_0$.