(1) Let $f \in L^p(\mathbb{R}^n)$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ so $\hat{f} \in \mathcal{S}'(\mathbb{R}^n)$ defined by $\langle \hat{f}, \phi \rangle := \langle f, \hat{\phi} \rangle$, $\phi \in \mathcal{S}$.

Note $f \mathbb{1}_{B_j(0)} \to f$ a.e. as $j \to \infty$, so for $\phi \in \mathcal{S}$,

$$\langle f \mathbb{1}_{B_j(0)}, \phi \rangle = \int_{B_j(0)} f\phi \, dx \xrightarrow[j\to\infty]{} \int_{\mathbb{R}^n} f\phi \, dx =$$

$\langle f, \phi \rangle$ by DCT ($|f \mathbb{1}_{B_j(0)} \phi| \leq |f||\phi| \leq |f|(1+|\cdot|)^{-n-1} \sup(1+|x|)^{n+1}|\phi| \in L^1$ for all $p \in [1,\infty]$),

But then $f \mathbb{1}_{B_j(0)} \to f$ in $\mathcal{S}'$ as $j \to \infty$ and so by $\mathcal{S}'$ continuity of $\mathcal{F}$ we get (since $f \mathbb{1}_{B_j(0)} \in L^1$ and we have consistency) that

$$g_j(\xi) = \int_{B_j(0)} f(x) e^{-i\xi\cdot x} \, dx \xrightarrow[j\to\infty]{} \hat{f} \text{ in } \mathcal{S}'.$$

Since $\hat{\delta}_0 = 1$ the differentiation rule gives $\widehat{\delta_0^{(k)}} = (i\xi)^k$, hence $\xi^k = (-i)^k \widehat{\delta_0^{(k)}}$ and so $\widehat{x^k} = (-i)^k \widehat{\widehat{\delta_0^{(k)}}} = (-i)^k 2\pi \widetilde{\delta_0^{(k)}} = i^k 2\pi \delta_0^{(k)}$ by FIF in $\mathcal{S}'$. But then $\int_{-j}^{j} x^k e^{-i\xi x} dx \xrightarrow[j\to\infty]{} 2\pi i^k \delta_0^{(k)}$.

②

A polynomial is in particular a tempered distribution so $\widehat{p(x)} \in \mathcal{S}'$ when $p(x) \in \mathbb{C}[x]$.

Now $\hat{\delta}_0 = 1$ so by differentiation rule

$$\widehat{p(x)} = \widehat{p(x)\hat{\delta}_0} = p(i\partial)\,\delta_0 .$$

When $p(x) = \sum_{|\alpha| \leq d} c_\alpha x^\alpha$ this can be written

$$\widehat{p(x)} = \sum_{|\alpha| \leq d} c_\alpha i^{|\alpha|} \partial^\alpha \delta_0 \in \operatorname{span}\{\partial^\alpha \delta_0 : \alpha \in \mathbb{N}_0^n\}.$$

Conversely, if $u \in \mathcal{S}'$ and $\operatorname{supp}(\hat{u}) \subseteq \{0\}$, then by result from B4.3

$$\hat{u} \in \operatorname{span}\{\partial^\alpha \delta_0 : \alpha \in \mathbb{N}_0^n\},$$

say $\hat{u} = \sum_{|\alpha| \leq d} c_\alpha \partial^\alpha \delta_0$ for some $d \in \mathbb{N}_0$, $c_\alpha \in \mathbb{C}$.

(Of course $\hat{u} = 0$ if $\operatorname{supp}(\hat{u}) = \emptyset$, so $u = 0$.)

By the differentiation rule

$$u = \sum_{|\alpha| \leq d} c_\alpha (ix)^\alpha \in \mathbb{C}[x].$$

(3) Clearly $e^x \in L^1_{loc} \subset \mathcal{D}'$. Assume $e^x \in \mathcal{S}'$ too and put $\chi_j = \rho * \mathbb{1}_{(-j-1, j+1)}$ $(j \in \mathbb{N})$ where ρ is the standard mollifier kernel on $\mathbb{R}$.

Clearly $\chi_j e^{-x} \rho * H \in \mathcal{D}$ (H = Heaviside's fct), $e^{-x} \rho * H \in \mathcal{S}$ and $\chi_j e^{-x} \rho * H \to e^{-x} \rho * H$ in $\mathcal{S}$ as $j \to \infty$: $S_{k,\ell}\left(e^{-x} \rho * H (1-\chi_j)\right) =$

$$\sup_x \left| x^k \left(e^{-x} \rho * H (1-\chi_j)\right)^{(\ell)} \right| \leq$$

$$\sup_x \left| x^k \sum_{s=0}^{\ell} \binom{\ell}{s} \left(e^{-x} \rho * H\right)^{(s)} (1-\chi_j)^{(\ell-s)} \right| \leq$$

$$\sup_x \left| x^k \left(e^{-x} \rho * H\right)^{(\ell)} (1-\chi_j) \right| + 2^\ell \sum_{s=0}^{\ell-1} \sup_x \left| x^k \left(e^{-x} \rho * H\right)^{(s)} \chi_j^{(\ell-s)} \right|$$

$$\leq \sup_{x \geq j} \left| x^k e^{-x} \max_{0 \leq s \leq \ell} |\rho^{(s)}| \, 2^\ell \right|$$

$$2^\ell \ell \sup_{x \geq j} \left| x^k e^{-x} \max_{0 \leq s \leq \ell} |\rho^{(s)}|^2 \right|$$

$$= (1+\ell) 2^\ell \max_{0 \leq s \leq \ell} |\rho^{(s)}|^2 \sup_{x \geq j} |x^k e^{-x}| \xrightarrow[j \to \infty]{} 0.$$

But then $\langle e^x, \chi_j e^{-x} \rho * H \rangle \to \langle e^x, e^{-x} \rho * H \rangle$ by $\mathcal{S}$ continuity, however,

$$\langle e^x, \chi_j e^{-x} \rho * H \rangle = \int_{-\infty}^{\infty} \chi_j \, \rho * H \, dx$$

$\geq j \to \infty$ ↯.

Clearly we have $e^{x+ie^x} \in C^\infty(\mathbb{R})$ with $(e^{ie^x})' = e^{ie^x} ie^x$. Now $-ie^{ie^x} \in L^\infty(\mathbb{R})$ so $-ie^{ie^x} \in \mathcal{S}'(\mathbb{R})$ and its distributional derivative is

$$\langle (-ie^{ie^x})', \varphi \rangle = \langle e^{x+ie^x}, \varphi \rangle$$

for $\varphi \in \mathcal{D}(\mathbb{R})$. It follows that $e^{x+ie^x} \in \mathcal{S}'(\mathbb{R})$ and we have for $\phi \in \mathcal{S}(\mathbb{R})$

$$\langle e^{x+ie^x}, \phi \rangle = \lim_{j \to \infty} \int_{-\infty}^{j} e^{x+ie^x} \phi(x) \, dx.$$

An improper integral since $e^{x+ie^x} \phi(x)$ need not be integrable on $\mathbb{R}$ when $\phi \in \mathcal{S}(\mathbb{R})$.

(4)

(a) For $u \in \mathcal{S}'(\mathbb{R}^n)$ the dilation rules say

$$\widehat{d_r u} = (\hat{u})_r \ , \quad \widehat{u_r} = d_r \hat{u} \quad \text{for } r>0.$$

If $d_r u = r^\alpha u$, then $d_r \hat{u} = \widehat{u_r} = \frac{1}{r^n} \widehat{d_{\frac{1}{r}} u} = \frac{1}{r^n} \widehat{\left(\frac{1}{r}\right)^\alpha u} = r^{-n-\alpha} \hat{u}$, and since this is true for any $r>0$, $\hat{u}$ is $-n-\alpha$ homogeneous. Likewise, if $d_r \hat{u} = r^{-n-\alpha} \hat{u}$, then $\widehat{d_r u} = (\hat{u})_r = \frac{1}{r^n} d_{\frac{1}{r}} \hat{u} = \frac{1}{r^n} \left(\frac{1}{r}\right)^{-n-\alpha} \hat{u}$

$$= r^\alpha \hat{u} = \widehat{r^\alpha u},$$

and so by FIF in $\mathcal{S}'$, $d_r u = r^\alpha u$. □

(b) Let $(\rho_\varepsilon)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}^n$. Recall that ρ_ε in particular is radial so that $\theta_* \rho_\varepsilon = \rho_\varepsilon$ for all $\theta \in O(n)$. Put $u_\varepsilon = \rho_\varepsilon * u$. Then u_ε is a moderate C^∞ function and $u_\varepsilon \xrightarrow[\varepsilon \to 0]{} u$ in $\mathcal{S}'(\mathbb{R}^n)$.

We check that $\theta_* u_\varepsilon = u_\varepsilon$ for $\theta \in O(n)$:

$$\theta_* u_\varepsilon(x) = u_\varepsilon(\theta x) = \langle u, \rho_\varepsilon(\theta x - \cdot) \rangle =$$

$$\langle \theta_*^{-1} u, \rho_\varepsilon(\theta x - \cdot) \rangle = \langle u, \rho_\varepsilon(\theta x - \theta(\cdot)) \rangle$$

$= \langle u, \varphi_\varepsilon(x - \cdot) \rangle = u_\varepsilon(x)$ for all $x \in \mathbb{R}^n$.

We may therefore define $f_\varepsilon : [0,\infty) \to \mathbb{C}$ by $f_\varepsilon(|x|) := u_\varepsilon(x)$, $|x| \geq 0$. Then f_ε is clearly a continuous function. Now we assume $u \in L^1_{loc}(\mathbb{R}^n)$ and note that for each $R > 0$, $u_\varepsilon \xrightarrow[\varepsilon \to 0]{} u$ in $L^1(B_R(0))$, so that (u_ε) is Cauchy in $L^1(B_R(0))$ as $\varepsilon \to 0$. Calculating in polar coordinates we find for $\varepsilon_1, \varepsilon_2 > 0$:

$$\| u_{\varepsilon_1} - u_{\varepsilon_2} \|_{L^1(B_R(0))} = \int_0^R \int_{\partial B_r(0)} |u_{\varepsilon_1}(x) - u_{\varepsilon_2}(x)| \, dS_x \, dr$$

$$= \omega_{n-1} \int_0^R |f_{\varepsilon_1}(r) - f_{\varepsilon_2}(r)| \, r^{n-1} \, dr.$$

Hence (f_ε) is Cauchy in $L^1((0,R), r^{n-1}dr)$ as $\varepsilon \to 0$, so by Riesz-Fischer it is convergent there: $f_\varepsilon \xrightarrow[\varepsilon \to 0]{} f^R$ in $L^1((0,R), r^{n-1}dr)$ for some $f^R \in L^1((0,R), r^{n-1}dr)$. It is clear that for $0 < S < R$ we have $f^R|_{(0,S)} = f^S$ as elements of $L^1((0,S), r^{n-1}dr)$, so there

exists $f \in L^1_{loc}([0,\infty), r^{n-1}dr)$ with the property $u(x) = f(|x|)$ for $\mathcal{L}^n$ a.e. $x \in \mathbb{R}^n$.
(A less precise argument is acceptable here.)

(c) A fundamental solution E to Δ on $\mathbb{R}^3$ is any $E \in \mathcal{D}'(\mathbb{R}^3)$ with $\Delta E = \delta_0$.

We seek $E \in \mathcal{S}'(\mathbb{R}^3)$ with $\Delta E = \delta_0$.
By Fourier transformation this is equivalent to $\hat{E} = -\frac{1}{|\xi|^2} \in L^1 + L^2$. By FIF in $\mathcal{S}'$, Riemann-Lebesgue and Plancherel:

$$E = \mathcal{F}^{-1}_{\xi \to x}\left(-\frac{1}{|\xi|^2}\right) \in C_0 + L^2 \subset L^1_{loc}.$$

Because $\xi \mapsto \frac{1}{|\xi|^2}$ is radial and (-2)-homogeneous, E is radial and (-1)-homogeneous and since $E \in L^1_{loc}$ it follows that E is $\frac{c}{|x|}$ for some $c \in \mathbb{R}$. In fact, $c \in \mathbb{R}$ since (by Q1); $E = \lim_{j \to \infty} (2\pi)^{-3} \int_{|\xi|<j} \frac{1}{|\xi|^2} e^{ix\cdot\xi} d\xi$

$$= \lim_{j\to\infty} (2\pi)^{-3} \int_{|\xi|<j} \frac{\cos(x\cdot\xi)}{|\xi|^2} d\xi \quad \text{in } \mathcal{S}'(\mathbb{R}^3).$$

5

(a) We have $\left|\frac{\sin(\xi)}{\xi}\right| \leq 1$ for all $\xi \in \mathbb{R}$ and so when $0 < s < t \leq 1$:

$$\left|\int_s^t \frac{\sin(\xi)}{\xi}\, d\xi\right| \leq t - s < 1.$$

In order to cover remaining cases we re= write using partial integration:

$$\int_s^t \frac{\sin \xi}{\xi}\, d\xi = \left[\frac{-\cos \xi}{\xi}\right]_s^t - \int_s^t \frac{\cos \xi}{\xi^2}\, d\xi .$$

If $0 < s \leq 1 < t$, then

$$\begin{aligned}
\left|\int_s^t \frac{\sin \xi}{\xi}\, d\xi\right| &= \left|\left(\int_s^1 + \int_1^t\right) \frac{\sin \xi}{\xi}\, d\xi\right| \\
&\leq 1 - s + \left|\cos 1 - \frac{\cos t}{t} - \int_1^t \frac{\cos \xi}{\xi^2}\, d\xi\right| \\
&< 1 + \cos 1 + \frac{|\cos t|}{t} + \int_1^t \frac{d\xi}{\xi^2} \\
&= 1 + \cos 1 + \frac{|\cos t|}{t} + 1 - \frac{1}{t} < 3.
\end{aligned}$$

If $1 < s < t$, then $\left|\int_s^t \frac{\sin \xi}{\xi}\, d\xi\right| \leq$

$$\left|\frac{\cos s}{s} - \frac{\cos t}{t}\right| + \int_s^t \frac{d\xi}{\xi^2} \leq \frac{1}{s} + \frac{1}{t} - \frac{1}{t} + \frac{1}{s} = \frac{2}{s} < 2.$$

In particular we get the required bound.

(b) Let $f \in L^1(\mathbb{R})$ be odd. Then

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx = \int_{-\infty}^{\infty} f(x)\left(\cos(x\xi) - i\sin(x\xi)\right)dx$$

$$\overset{f \text{ odd}}{=} -2i \int_0^\infty f(x)\sin(x\xi)\,dx .$$

Now for $0<s<t$ the function

$$(x,\xi) \mapsto f(x)\,\frac{\sin(x\xi)}{\xi}$$

is integrable over $(0,\infty)\times(s,t)$, so by Fubini

$$\int_s^t \frac{\hat{f}(\xi)}{\xi}\,d\xi = -2i \int_s^t \int_0^\infty f(x)\,\frac{\sin(x\xi)}{\xi}\,dx\,d\xi$$

$$= -2i \int_0^\infty f(x) \int_s^t \frac{\sin(x\xi)}{\xi}\,d\xi\,dx$$

$$\overset{\eta = x\xi}{=} -2i \int_0^\infty f(x) \int_{sx}^{tx} \frac{\sin(\eta)}{\eta}\,d\eta\,dx ,$$

and consequently

$$\left|\int_s^t \frac{\hat{f}(\xi)}{\xi}\,d\xi\right| \leq 2\int_0^\infty |f(x)| \cdot \left|\int_{sx}^{tx} \frac{\sin(\eta)}{\eta}\,d\eta\right| dx$$

$$\leq 8 \int_0^\infty |f(x)|\,dx = 4\|f\|_1 .$$

(c) Assume $g \in C_0(\mathbb{R})$ is odd and that $g(\xi) = \dfrac{1}{\log \xi}$ for $\xi \geqslant 2$. Since for $2 \leq s < t$

$$\int_s^t \frac{g(\xi)}{\xi}\,d\xi = \log \frac{\log t}{\log s}$$

is unbounded conclusion follows.

(6) Take $h = \mathcal{F}^{-1}(\rho_{\frac{1}{2}} * \mathbb{1}_{B_{\frac{3}{2}}(0)})$. Then

$h \in \mathcal{S}(\mathbb{R}^n)$ with $\hat{h} = 1$ on $B_1(0)$, $\hat{h} = 0$ off $B_2(0)$.

If $f \in L^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(\hat{f}) \subseteq \overline{B_r(0)}$, then, by FIF in $\mathcal{S}'$, $f = \mathcal{F}^{-1}\hat{f} = (2\pi)^{-n}\mathcal{F}\check{\hat{f}}$ is a moderate C^∞ function (and Paley-Wiener tells us much more). Since $d_{\frac{1}{r}}\hat{h} = 1$ on $\overline{B_r(0)}$ also $\hat{f} = \hat{f}\, d_{\frac{1}{r}}\hat{h}$, so by the convolution rule + dilation $f = f * h_{\frac{1}{r}}$.

For $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha f = (\partial^\alpha h_{\frac{1}{r}}) * f =$

$(\frac{1}{r})^{-|\alpha|}(\partial^\alpha h)_{\frac{1}{r}} * f$, hence

$$\|\partial^\alpha f\|_\infty = r^{|\alpha|}\|(\partial^\alpha h)_{\frac{1}{r}} * f\|_\infty$$

$$\leq r^{|\alpha|}\|(\partial^\alpha h)_{\frac{1}{r}}\|_1 \|f\|_\infty$$

$$= r^{|\alpha|}\|\partial^\alpha h\|_1 \cdot \|f\|_\infty .$$

(7) $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\Delta u \in L^2(\mathbb{R}^n)$.

(a) By the differentiation rule:

$$\widehat{\partial_j \partial_k \Delta u} = \xi_j \xi_k \underbrace{(|\xi|^2 \hat{u})}_{\text{an } L^2 \text{ function by Plancherel}} = |\xi|^2 \frac{\xi_j \xi_k}{|\xi|^2} (|\xi|^2 \hat{u})$$

Note: The issue is that multiplication isn't associative here.

$$= \mathcal{F}\left(\Delta \mathcal{F}^{-1}\left(\frac{\xi_j \xi_k}{|\xi|^2} \widehat{\Delta u}\right)\right).$$

But $\partial_j \partial_k \Delta u = \Delta \partial_j \partial_k u$ so also, by FIF in $\mathcal{S}'$,

$$\Delta \partial_j \partial_k u = \Delta \mathcal{F}^{-1}\left(\frac{\xi_j \xi_k}{|\xi|^2} \widehat{\Delta u}\right),$$

hence

$$\Delta\left(\partial_j \partial_k u - \mathcal{F}^{-1}\left(\frac{\xi_j \xi_k}{|\xi|^2} \widehat{\Delta u}\right)\right) = 0.$$

Now if $H \in \mathcal{S}'(\mathbb{R}^n)$ and $\Delta H = 0$, then $\operatorname{supp} \hat{H} \subseteq \{0\}$, and so H is a polynomial (by Q2) with $\Delta H = 0$, that is, H is a harmonic polynomial. But then we find for each $1 \leq j,k \leq n$ a harmonic polynomial $P_{j,k}$ so

$$\partial_j \partial_k u - P_{j,k} = \mathcal{F}^{-1}\left(\frac{\xi_j \xi_k}{|\xi|^2} \widehat{\Delta u}\right)$$

and the conclusion now follows as in the proof of Prop. 1.74.

(t) We have assumed that $\mathcal{F}^{-1}\left(\frac{\xi_j\xi_k}{|\xi|^2}\widehat{\Delta u}\right) = \partial_j\partial_k m$ for some $m \in \mathcal{S}'(\mathbb{R}^n)$. But then

$$P_{j,k} = \partial_j\partial_k(u-m), \quad 1 \leq j,k \leq n,$$

and it follows that $u-m$ is a polynomial.