Problem 1: (a) Write $\zeta = \xi + \mathrm{i}\eta$ and note that when $\eta > 0$

$$\begin{aligned}\left|f(x)\mathrm{e}^{-\mathrm{i}\zeta x}\right| =& \left|f(x)\right|\mathrm{e}^{\eta x}\\ \leq& \tfrac{1}{2}\left(|f(x)|^2 + \mathrm{e}^{2\eta x}\right) \in \mathrm{L}^1(-\infty, 0),\end{aligned}$$

so $F(\zeta)$ is well-defined for $\zeta \in H$. Fix $\zeta \in H$ and consider for $z \in \mathbb{C}$ so $z \neq 0$ and $z + \zeta \in H$ the difference quotient

$$\tfrac{F(\zeta+z)-F(\zeta)}{z} = \int_{-\infty}^{0} f(x)\mathrm{e}^{-\mathrm{i}\zeta x}\tfrac{\mathrm{e}^{-\mathrm{i}zx}-1}{z}\,\mathrm{d}x.$$

We want to apply Lebesgue's dominated convergence theorem. Write $z = \alpha + \mathrm{i}\beta$ and use the fundamental theorem of calculus to estimate

$$\left|\tfrac{\mathrm{e}^{-\mathrm{i}zx}-1}{z}\right| \leq \int_0^1 \mathrm{e}^{\beta xt}\,\mathrm{d}t|x|,$$

hence for a.e. $x < 0$:

$$\begin{aligned}\left|f(x)\mathrm{e}^{-\mathrm{i}\zeta x}\tfrac{\mathrm{e}^{-\mathrm{i}zx}-1}{z}\right| \leq& \tfrac{1}{2}|f(x)|^2 + \tfrac{1}{2}\mathrm{e}^{2\eta x}\left(\int_0^1 \mathrm{e}^{\beta xt}\,\mathrm{d}t\right)^2 x^2\\ \leq& \tfrac{1}{2}|f(x)|^2 + \tfrac{1}{2}\mathrm{e}^{2\eta x}\mathrm{e}^{-2|\beta|x}x^2\\ =& \tfrac{1}{2}|f(x)|^2 + \tfrac{x^2}{2}\mathrm{e}^{2(\eta-|\beta|)x}.\end{aligned}$$

When $|\beta| < \eta$ the latter is integrable over $(-\infty, 0)$, hence by DCT, $f(x)\big(-\mathrm{i}x\big)\mathrm{e}^{-\mathrm{i}\zeta x} \in \mathrm{L}^1(\mathbb{R})$ and

$$\tfrac{F(\zeta+z)-F(\zeta)}{z} \to \int_{-\infty}^{0} f(x)\big(-\mathrm{i}x\big)\mathrm{e}^{-\mathrm{i}\zeta x}\,\mathrm{d}x \text{ as } z \to 0.$$

Thus $F\colon H \to \mathbb{C}$ is complex differentiable at ζ and since $\zeta \in H$ was arbitrary F is holomorphic. *[Alternatively you can use Fubini's and Morera's theorems.]*
Now for $\eta > 0$ we have $x \mapsto f(x)\mathrm{e}^{\eta x}$ is square integrable (and integrable) over $\mathbb{R}$ (recall: $f \equiv 0$ on $(0, \infty)$) so by Plancherel's theorem

$$\mathcal{F}_{x\to\xi}\big(f(x)\mathrm{e}^{\eta x}\big) = \lim_{j,\,k\to\infty}\int_{-j}^{k} f(x)\mathrm{e}^{\eta x}\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x$$

with convergence in the sense of $\mathrm{L}^2(\mathbb{R})$ and by Parseval's identity

$$\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathcal{F}_{x\to\xi}\big(f(x)\mathrm{e}^{\eta x}\big)\right|^2\mathrm{d}\xi = \int_{-\infty}^{\infty}|f(x)\mathrm{e}^{\eta x}|^2\,\mathrm{d}x.$$

Here the left-hand side, L, can be rewritten: first note that $x \mapsto f(x)\mathrm{e}^{\eta x}\mathrm{e}^{-\mathrm{i}\xi x}$ is integrable over $\mathbb{R}$ so by Lebesgue's DCT

$$L = \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{0} f(x)\mathrm{e}^{\eta x}\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x\right|^2\mathrm{d}\xi = \tfrac{1}{2\pi}\int_{-\infty}^{\infty}|F(\xi+\mathrm{i}\eta)|^2.$$

Combining this with

$$\int_{-\infty}^{\infty}|f(x)\mathrm{e}^{\eta x}|^2\,\mathrm{d}x = \int_{\infty}^{0}|f(x)|^2\mathrm{e}^{2\eta x}\,\mathrm{d}x \leq \|f\|_2^2$$

when $\eta > 0$ we conclude with the required bound for F. Finally, because $f(x)\mathrm{e}^{\eta x} \to f(x)$ in $\mathrm{L}^2(\mathbb{R})$ as $\eta \searrow 0$ (for instance from Lebesgue's monotone convergence theorem), Plancherel's theorem yields $F(\cdot + \mathrm{i}\eta) \to \widehat{f}$ in $\mathrm{L}^2(\mathbb{R})$ as $\searrow 0$.

(b) (*Optional*) Put $F_\eta(\xi) := F(\xi + \mathrm{i}\eta)$, $\xi \in \mathbb{R}$, for each $\eta > 0$. Then $F_\eta \in \mathrm{L}^2(\mathbb{R})$ and so by Plancherel's theorem and Parseval's identity $\widehat{F_\eta} \in \mathrm{L}^2(\mathbb{R})$ and $\|\widehat{F_\eta}\|_2 = \sqrt{2\pi}\|F_\eta\|_2$. By the Fourier inversion formula for tempered distributions we have

$$F_\eta = \tfrac{1}{2\pi}\widetilde{\widehat{\widehat{F_\eta}}} = \mathcal{F}\big(\tfrac{1}{2\pi}\widetilde{\widehat{F_\eta}}\big),$$

where we used that the Fourier transform commutes with the operation $\widetilde{(\cdot)}$. Consider

$$\begin{aligned}\widetilde{\widehat{F_\eta}}(x) =& \int_{-\infty}^{\infty} F_\eta(\xi)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi \\ =& \mathrm{e}^{\eta x}\int_{-\infty}^{\infty} F(\xi+\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}x(\xi+\mathrm{i}\eta)}\,\mathrm{d}\xi,\end{aligned}$$

where the integrals must be understood to converge in the L^2 sense (according to Plancherel). Here the function $H \ni \zeta \mapsto F(\zeta)\mathrm{e}^{\mathrm{i}x\zeta}$ is holomorphic for each fixed $x \in \mathbb{R}$, so if for a fixed $\eta \in (0,1)\cup(1,\infty)$ and positive numbers r, $s > 0$ we denote by $\Gamma_{r,s}$ the rectangle with corners $s+\mathrm{i}$, $s+\mathrm{i}\eta$, $-r+\mathrm{i}\eta$, $-r+\mathrm{i}$ traversed counter clockwise, then we have by Cauchy's theorem,

$$\int_{\Gamma_{r,s}} F(\zeta)\mathrm{e}^{\mathrm{i}x\zeta}\,\mathrm{d}\zeta = 0.$$

In order to see that we can *choose* r, $s \to \infty$ so that the contour integrals over corresponding vertical parts of $\Gamma_{r,s}$ tend to 0 we must use the assumed L^2 bound: First we have by Hölder's inequality

$$\begin{aligned}\left|\int_1^\eta F(\xi+\mathrm{i}t)\,\mathrm{e}^{\mathrm{i}x(\xi+\mathrm{i}t)}\mathrm{i}\mathrm{d}t\right|^2 \quad&\leq\quad |\eta-1|\left|\int_1^\eta |F(\xi+\mathrm{i}t)|^2\mathrm{e}^{-2xt}\,\mathrm{d}t\right| \\ &\leq\quad \eta\mathrm{e}^{2|x|(1+\eta)}\int_0^{1+\eta}|F(\xi+\mathrm{i}t)|^2\,\mathrm{d}t.\end{aligned}\tag{0.1}$$

By Tonelli's theorem and our assumption

$$\begin{aligned}\int_{-\infty}^{\infty}\int_0^{1+\eta}|F(\xi+\mathrm{i}t)|^2\,\mathrm{d}t\,\mathrm{d}\xi =& \int_0^{1+\eta}\int_{-\infty}^{\infty}|F(\xi+\mathrm{i}t)|^2\,\mathrm{d}\xi\,\mathrm{d}t \\ \leq& \sup_{t>0}\int_{-\infty}^{\infty}|F(\xi+\mathrm{i}t)|^2\,\mathrm{d}\xi(1+\eta) < \infty.\end{aligned}$$

We can therefore find values $r = r_j$, $s = s_j \to \infty$ so

$$\int_0^{1+\eta}|F(-r_j+\mathrm{i}t)|^2\,\mathrm{d}t \to 0 \;\text{ and }\; \int_0^{1+\eta}|F(s_j+\mathrm{i}t)|^2\,\mathrm{d}t \to 0 \;\text{ as }\; j\to\infty$$

as otherwise the above integrals would be divergent (note that we cannot in general be sure that we have this for all choices of r, $s \to \infty$). It follows from (0.1) that the contour integrals over the corresponding vertical parts of Γ_{r_j,s_j} tend to 0 as $j \to \infty$. Thus

$$\int_{-r_j}^{s_j} F(\xi+\mathrm{i})\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi\,\mathrm{e}^{-x} = \int_{-r_j}^{s_j} F(\xi+\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi\,\mathrm{e}^{-\eta x} + \mathrm{o}(1)$$

as $j \to \infty$ for each fixed $x \in \mathbb{R}$. Now Plancherel's theorem ensures that the two integrals above both converge in the sense of L^2 as functions of $x \in \mathbb{R}$ when $j \to \infty$ and by properties of L^2 convergence it follows that for a suitable subsequence, say (j_k) of (j), we have convergence pointwise almost everywhere. Consequently

$$\widetilde{\widehat{F_1}}(x)\mathrm{e}^{-x} = \widetilde{\widehat{F_\eta}}(x)\mathrm{e}^{-\eta x}$$

holds for almost every $x \in \mathbb{R}$, say for all $x \in \mathbb{R} \setminus N_\eta$, where $\mathcal{L}^1(N_\eta) = 0$. Define

$$f(x) = \tfrac{1}{2\pi}\widetilde{\widehat{F_1}}(x)\mathrm{e}^{-x}.$$

This is a measurable function and, provided we have chosen a representative for the L^2 function $\widetilde{\widehat{F_1}}$ that is consistent with the above identity, we have

$$\widetilde{\widehat{F_\eta}}(x) = 2\pi f(x)\mathrm{e}^{\eta x}$$

for $x \in \mathbb{R} \setminus N_\eta$. The left-hand side is in L^2, hence so is the right-hand side and so we get from Parseval's identity and our assumption

$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{e}^{2\eta x}\,\mathrm{d}x = \tfrac{1}{(2\pi)^2}\|\widetilde{\widehat{F_\eta}}\|_2^2 = \tfrac{1}{2\pi}\|F_\eta\|_2^2 \leq m < \infty$$

for a constant m (independent of $\eta > 0$). It follows that $f(x) = 0$ for a.e. $x > 0$: Fix $\varepsilon > 0$ and put $E = \bigl\{x > \varepsilon : |f(x)| \geq \varepsilon\bigr\}$. Then E is a measurable set and $|f(x)|^2\mathrm{e}^{2\eta x} \geq \varepsilon^2\mathrm{e}^{2\eta\varepsilon}$ a.e. on E, so

$$\varepsilon^2\mathrm{e}^{2\varepsilon\eta}\mathcal{L}^1(E) \leq \int_E |f(x)|^2\mathrm{e}^{2\eta x}\,\mathrm{d}x \leq m$$

for all $\eta > 0$, which is ony possible if $\mathcal{L}^1(E) = 0$. Since $\varepsilon > 0$ was arbitrary it follows that $f = 0$ a.e. on $(0, \infty)$ as asserted. By Lebesgue's monotone convergence theorem we find

$$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \int_{-\infty}^{0} |f(x)|^2\,\mathrm{d}x = \sup_{\eta>0}\int_{-\infty}^{0} |f(x)|^2\mathrm{e}^{2\eta x}\,\mathrm{d}x \leq m < \infty,$$

so $f \in \mathrm{L}^2(\mathbb{R})$. Finally, we conclude with $F_\eta \to \widehat{f}$ in L^2 as $\eta \searrow 0$ by use of Plancherel's and Fourier's inversion theorems and the above identities.

Problem 2: (a) f is piecewise C^1 so its distributional derivative is

$$f'(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ -1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\widehat{f'}(\xi) = \frac{2}{\mathrm{i}\xi}\bigl(\cos\xi - 1\bigr)$, so by the differentiation rule,

$$\widehat{f}(\xi) = 2\frac{1-\cos\xi}{\xi^2},$$

and using the double-angle formula, $\cos\xi = \cos^2\bigl(\xi/2\bigr) - \sin^2\bigl(\xi/2\bigr)$, and $1 - \cos^2\bigl(\xi/2\bigr) = \sin^2\bigl(\xi/2\bigr)$ we conclude that $\widehat{f}(\xi) = \operatorname{sinc}^2\bigl(\xi/2\bigr)$. The Poisson summation formula can be stated as

$$\sum_{k\in\mathbb{Z}} \mathrm{e}^{-2\pi\mathrm{i}kx} = \sum_{k\in\mathbb{Z}} \delta_k$$

with convergence of the series in $\mathcal{S}'(\mathbb{R})$. Note that the distribution on LHS is continuous with respect to the norm $\phi \mapsto \overline{S}_{2,0}(\widehat{\phi})$ and the one on the RHS with respect to the norm $\overline{S}_{2,0}$. By inspection we see that $\overline{S}_{2,0}(f) = 1$, $\overline{S}_{2,0}(\widehat{f}) = 4$, in particular that both are finite, so by approximation we may evaluate the above identity at f, and what is more convenient here, on $x \mapsto f(x)\mathrm{e}^{-\mathrm{i}hx}$ for a parameter $h \in \mathbb{R}$. By the translation rule

$$\widehat{f}(\xi + h) = \mathcal{F}_{x\to\xi}\bigl(f(x)\mathrm{e}^{-\mathrm{i}hx}\bigr),$$

whereby we find

$$\sum_{k\in\mathbb{Z}} \operatorname{sinc}^2(2\pi k + h) = \sum_{k\in\mathbb{Z}} \bigl(1 - |k|\bigr)^+\mathrm{e}^{-\mathrm{i}hk} = 1.$$

Here we have using the addition formula and assuming $h \in \mathbb{R} \setminus 2\pi\mathbb{Z}$:

$$\begin{aligned}
\operatorname{sinc}^2(2\pi k + h) =& \frac{\sin^2\left(\pi k + \frac{h}{2}\right)}{\left(\pi k + \frac{h}{2}\right)^2} \\
=& \frac{\sin^2\left(\frac{h}{2}\right)}{\pi^2\left(k + \frac{h}{2\pi}\right)^2} \\
=& \frac{\sin^2\left(\frac{h}{2}\right)}{\pi^2} \frac{1}{\left(k + \frac{h}{2\pi}\right)^2},
\end{aligned}$$

and so

$$\sum_{k \in \mathbb{Z}} \frac{1}{\left(k + \frac{h}{2\pi}\right)^2} = \frac{\pi^2}{\sin^2\left(\frac{h}{2}\right)}.$$

The desired formula follows if we take $x = \frac{h}{2\pi} \in \mathbb{R} \setminus \mathbb{Z}$.
(b) Note that for $x \in \mathbb{R} \setminus \mathbb{Z}$:

$$\sum_{n=-N}^{N} \frac{1}{n+x} = \frac{1}{x} - \sum_{n=1}^{N} \frac{2x}{n^2 - x^2},$$

and so using Weierstrass' M-test we see that the LHS converges locally uniformly on $\mathbb{R} \setminus \mathbb{Z}$, and so defines a continuous function there. Denote its restriction to $(0,1)$ by T. Clearly $T \in \mathcal{D}'(0,1)$ and by $\mathcal{D}'$ continuity of differentiation we get

$$\begin{aligned}
T' &= \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{-1}{(n+x)^2} \\
c &= -\sum_{k \in \mathbb{Z}} \frac{1}{(k+x)^2}.
\end{aligned}$$

Therefore we have according to (a), $T' = -\frac{\pi^2}{\sin^2(\pi x)} = \frac{\mathrm{d}}{\mathrm{d}x} \pi \cot \pi x$ in $\mathcal{D}'(0,1)$, and so by the constancy theorem, $T = \pi \cot \pi x + c$ for some constant $c \in \mathbb{C}$. Evaluating the identity at $x = 1/2$ we see that $c = 0$ and so that the identity holds on $(0,1)$. However, the two sides are both 1 periodic and so the desired identity then clearly holds on all of $\mathbb{R} \setminus \mathbb{Z}$.
(c) The argument with the Weierstrass M-test used in (b) also easily gives that $\sum_{n=-N}^{N} \frac{1}{n+z}$ converges locally uniformly in $z \in \mathbb{C} \setminus \mathbb{Z}$ as $N \to \infty$. It therefore follows from Morera's theorem that $z \mapsto \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z+n}$ is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$. Since also $\pi \cot \pi z$ is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$ and the two functions agree on $\mathbb{R} \setminus \mathbb{Z}$ it follows from the identity theorem that

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{x+z} = \pi \cot \pi z \tag{0.2}$$

holds for all $z \in \mathbb{C} \setminus \mathbb{Z}$. In order to prove Lipschitz's formula we proceed by induction on k. However, we first note that when $k \geq 2$ we have for each $z \in H$ that

$$\left|\mathrm{e}^{2\pi \mathrm{i} z}\right| = \mathrm{e}^{-2\pi \operatorname{Im}(z)} < 1,$$

so the series on the right-hand side of the Lipschitz formula converges locally uniformly in $z \in H$. It is clear that the series on the left-hand side also converges locally uniformly in $z \in H$ when $k \geq 2$. Both sides therefore in particular define regular distributions that can be differentiated distributionally term-by-term. This is also true of (0.2) that will be our starting point: Indeed expressing cot in terms of complex exponentials and recognizing a geometric series we find

$$\pi \cot \pi z = -\pi \mathrm{i} \frac{1+\mathrm{e}^{2\pi \mathrm{i} z}}{1-\mathrm{e}^{2\pi \mathrm{i} z}} = -\pi \mathrm{i} \left(1 + 2 \sum_{n=1}^{\infty} \mathrm{e}^{2\pi \mathrm{i} n z} \right),$$

and hence

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{n+z} = -\pi \mathrm{i} \left(1 + 2 \sum_{n=1}^{\infty} \mathrm{e}^{2\pi \mathrm{i} n z} \right)$$

locally uniformly in $z \in H$. Differentiating with respect to z in the sense of distributions (precisely we apply the Cauchy-Riemann differential operator $\frac{\partial}{\partial z}$) we get Lipschitz's formula for $k = 2$ in the sense of distributions on H. But since both sides are regular distributions, in fact represented by holomorphic functions, we have established the formula for this special case. Now assume that Lipschitz's formula holds for some $k \geq 2$. Differentiating it we see that the formula also holds for $k+1$ in the sense of distributions on H. But again both sides are represented by holomorphic functions so the formula holds also for all $z \in H$. The Lipschitz formula therefore follows by induction on $k \geq 2$.

Problem 3: (*Optional*) The function $g \colon \mathbb{R} \to \mathbb{C}$ is clearly piecewise C^{∞} with jump discontinuities at points of $2\pi\mathbb{Z}$. In particular we record that g is locally square integrable, so Plancherel's theorem for Fourier series applies and we have

$$g(x) = \sum_{n \in \mathbb{Z}} c_n \mathrm{e}^{\mathrm{i}nx} \quad \text{in } \mathrm{L}^2.$$

We calculate the Fourier coefficients:

$$\begin{aligned} c_n = \tfrac{1}{2\pi} \int_0^{2\pi} g(x) \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x &= \tfrac{1}{2 \sin \pi\alpha} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\pi\alpha - \mathrm{i}(n+\alpha)x} \, \mathrm{d}x \\ &= \frac{\mathrm{e}^{\mathrm{i}\pi\alpha}}{2 \sin \pi\alpha} \frac{1 - \mathrm{e}^{-\mathrm{i}(n+\alpha)2\pi}}{\mathrm{i}(n+\alpha)} \\ &= \frac{1}{n+\alpha}, \end{aligned}$$

where we skipped some straight forward routine calculation in the last line. Consequently we have

$$g(x) = \sum_{n \in \mathbb{Z}} \frac{1}{n+\alpha} \mathrm{e}^{\mathrm{i}nx}$$

where the doubly infinite series converges in L^2. If we use Parseval's identity for Fourier series we recover (1) on the problem sheet:

$$\begin{aligned} \frac{\pi^2}{\sin^2 \pi\alpha} &= \tfrac{1}{2\pi} \int_0^{2\pi} |g(x)|^2 \, \mathrm{d}x \\ &= \sum_{n \in \mathbb{Z}} |c_n|^2 \\ &= \sum_{n \in \mathbb{Z}} \tfrac{1}{(n+\alpha)^2} \end{aligned}$$

for $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. In order to deduce (2) on the problem sheet we would have liked to take $x = 0$ in the Fourier expansion of g. However, this is not permitted by the theory we have developed so far since g has a jump discontinuity at $x = 0$:

$$g(0) = \tfrac{\pi}{\sin \pi\alpha} \mathrm{e}^{\mathrm{i}\pi\alpha} \neq \tfrac{\pi}{\sin \pi\alpha} \mathrm{e}^{-\mathrm{i}\pi\alpha} = g(2\pi-)$$

because $\alpha \in \mathbb{Z}$.

Problem 4: (a) That the periodisation $P\varphi$ is a 2π periodic C^∞ function follows as in the lecture notes page 50: Fix $n \in \mathbb{N}_0$ and note that the derivative $\varphi^{(n)}$ is again a Schwartz test function on $\mathbb{R}$, so that by prelims analysis it suffices to prove that the series defining $P\varphi$ is locally uniformly convergent on $\mathbb{R}$. For that we note that when $|x| \leq \pi$ and $k \in \mathbb{Z} \setminus \{0\}$:

$$\begin{aligned}\bigl|\varphi(x-2\pi k)\bigr| = \frac{1+(x-2\pi k)^2}{1+(x-2\pi k)^2}\bigl|\varphi(x-2\pi k)\bigr| &\leq \frac{2}{1+(x-2\pi k)^2}\overline{S}_{2,0}(\varphi)\\ &\leq \frac{2}{1+(2\pi|k|-\pi)^2}\overline{S}_{2,0}(\varphi),\end{aligned}$$

hence the series converges uniformly in $x \in [-\pi, \pi]$ by Weierstrass' M-test, and so $P\varphi$ is a 2π-periodic C^∞ function. Next, for each $k \in \mathbb{Z}$ we calculate (using uniform convergence to justify swapping order of integration and summation):

$$\begin{aligned}c_k = \tfrac{1}{2\pi}\int_0^{2\pi} P\varphi(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x &= \sum_{n\in\mathbb{Z}} \tfrac{1}{2\pi}\int_0^{2\pi} \varphi(x+2\pi k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\\ &= \sum_{n\in\mathbb{Z}} \tfrac{1}{2\pi}\int_{2\pi n}^{2\pi(n+1)} \varphi(x)\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}2\pi n}\,\mathrm{d}x\\ &= \sum_{n\in\mathbb{Z}} \tfrac{1}{2\pi}\int_{2\pi n}^{2\pi(n+1)} \varphi(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \tfrac{1}{2\pi}\widehat{\varphi}(k),\end{aligned}$$

and so

$$\sum_{k\in\mathbb{Z}} \tfrac{1}{2\pi}\widehat{\varphi}(k)\mathrm{e}^{\mathrm{i}kx} = \lim_{n\to\infty}\sum_{k=-n}^{n} c_k\mathrm{e}^{\mathrm{i}kx}$$

holds for each $x \in \mathbb{R}$ (and in fact it is not difficult to see that the convergence is uniform, and even much better). Now note that

$$\sum_{k=-n}^{n} c_k\mathrm{e}^{\mathrm{i}kx} = \int_0^{2\pi} P\varphi(x)D_n(x-y)\,\mathrm{d}y,$$

where

$$D_n(t) = \sum_{k=-n}^{n} \tfrac{1}{2\pi}\mathrm{e}^{\mathrm{i}kt}$$

is *Dirichlet's kernel.* For $t \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ we have by summing a geometric series and then rewriting in terms of sine:

$$D_n(t) = \tfrac{1}{2\pi}\frac{1-\mathrm{e}^{\mathrm{i}(2n+1)t}}{1-\mathrm{e}^{\mathrm{i}t}} = \tfrac{1}{2\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}.$$

We check that

$$\int_0^{2\pi} D_n(t)\,\mathrm{d}t = 1$$

and that, for each $\delta \in (0, \pi)$,

$$\int_\delta^{2\pi-\delta} D_n(t)\,\mathrm{d}t \to 0 \text{ as } n \to \infty$$

by the Riemann-Lebesgue lemma (note $1/\sin(t/2)$ is integrable over $(\delta, 2\pi-\delta)$). Put $L = \max\bigl|(P\varphi)'\bigr|$ and note that

$$\bigl|P\varphi(x) - P\varphi(x-t)\bigr| \leq L\min_{k\in\mathbb{Z}}|t-2\pi k|$$

holds for all $x,\, t \in \mathbb{R}$. For $\delta \in (0,\pi)$ we have

$$\begin{aligned}
\left| P\varphi(x) - \int_0^{2\pi} P\varphi(x-t) D_n(t)\,\mathrm{d}t \right| &= \left| \int_0^{2\pi} \big(P\varphi(x) - P\varphi(x-t)\big) D_n(t)\,\mathrm{d}t \right| \\
&\leq \left| \int_\delta^{2\pi-\delta} \big(P\varphi(x) - P\varphi(x-t)\big) D_n(t)\,\mathrm{d}t \right| \\
&\quad + \left| \left(\int_0^\delta + \int_{2\pi-\delta}^{2\pi} \right) \big(P\varphi(x) - P\varphi(x-t)\big) D_n(t)\,\mathrm{d}t \right| \\
&\leq \left| \int_\delta^{2\pi-\delta} \big(P\varphi(x) - P\varphi(x-t)\big) D_n(t)\,\mathrm{d}t \right| \\
&\quad + 4L \int_0^\delta \frac{t/2}{\sin(t/2)}\,\mathrm{d}t.
\end{aligned}$$

For fixed δ the first integral tends to 0 by the Riemann-Lebesgue lemma:

$$I := \int_\delta^{2\pi-\delta} \big(P\varphi(x) - P\varphi(x-t)\big) D_n(t)\,\mathrm{d}t \to 0 \text{ as } n \to \infty,$$

whereas

$$II := 4L \int_0^\delta \frac{t/2}{\sin(t/2)}\,\mathrm{d}t \to 0 \text{ as } \delta \searrow 0.$$

Thus given $\varepsilon > 0$ we take $\delta \in (0,\pi)$ so $II < \varepsilon$ and then for this fixed δ we obtain

$$\limsup_{n\to\infty} \left| P\varphi(x) - \int_0^{2\pi} P\varphi(x-t) D_n(t)\,\mathrm{d}t \right| \leq \varepsilon.$$

Consequently we have shown that $P\varphi(x) = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi}\widehat{\varphi}(k)\mathrm{e}^{\mathrm{i}kx}$ for all $x \in \mathbb{R}$. Taking $x=0$ we deduce the Poisson summation formula.

(b) If $G(x) = \mathrm{e}^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$, then $\widehat{G}(\xi) = \sqrt{2\pi}\mathrm{e}^{-\frac{\xi^2}{2}}$ (see Lemma 1.38 in the lecture notes) and so for $t>0$ we have $d_{\sqrt{2t}}G(x) = \mathrm{e}^{-tx^2}$, hence by the dilation rule

$$\widehat{d_{\sqrt{2t}}G}(\xi) = \big(\widehat{G}\big)_{\sqrt{2t}}(\xi) = \sqrt{\frac{\pi}{t}}\mathrm{e}^{-\frac{\xi^2}{4t}}.$$

The Poisson summation formula applied to $\varphi = d_{\sqrt{2t}}G$ now gives the desired formula.

Problem 5: If $\varphi \in \mathcal{S}(\mathbb{R})$, then also $\widehat{\varphi} \in \mathcal{S}(\mathbb{R})$ and the continuity of the Fourier transform is most conveniently expressed through the Fourier bounds: for $k,\, l \in \mathbb{N}_0$ there exists a constant $c = c_{k,l}$ so

$$\overline{S}_{k,l}(\widehat{\varphi}) \leq c\overline{S}_{l+2,k}(\varphi).$$

Thus we have in particular for $m \in \mathbb{N}$ and $\xi \in \mathbb{R} \setminus \{0\}$ that

$$\begin{aligned}
\big|\widehat{\varphi}\big| = \frac{|\xi|^{m+1}}{|\xi|^{m+1}}\big|\widehat{\varphi}(\xi)\big| &\leq \frac{\overline{S}_{m+1,0}(\widehat{\varphi})}{|\xi|^{m+1}} \\
&\leq \frac{c\overline{S}_{2,m+1}(\varphi)}{|\xi|^{m+1}}.
\end{aligned}$$

For $N > 0$ we apply the Poisson summation formula to the L^1 dilation $\varphi_{2\pi N}$ whereby we get, using also the dilation rule,

$$\begin{aligned}\tfrac{1}{N}\sum_{k\in\mathbb{Z}}\varphi\big(\tfrac{k}{N}\big) &= \sum_{k\in\mathbb{Z}}\widehat{\varphi}(2\pi Nk)\\ &= \widehat{\varphi}(0)+\sum_{k\neq 0}\widehat{\varphi}(2\pi Nk),\end{aligned}$$

hence

$$\int_{-\infty}^{\infty}\varphi(x)\,\mathrm{d}x = \tfrac{1}{N}\sum_{k\in\mathbb{Z}}\varphi\big(\tfrac{k}{N}\big)+R_N,$$

where

$$\begin{aligned}|R_N| = \left|-\sum_{k\neq 0}\widehat{\varphi}(2\pi Nk)\right| &\leq \sum_{k\neq 0}\frac{c\overline{S}_{2,m+1}(\varphi)}{|2\pi Nk|^{m+1}}\\ &= 2c\overline{S}_{2,m+1}(\varphi)\sum_{k=1}^{\infty}\frac{1}{(2\pi Nk)^{m+1}}\\ &\leq \frac{c\overline{S}_{2,m+1}(\varphi)}{N^{m+1}}.\end{aligned}$$

Problem 6: (a) We have for $n \neq 0$ using 2π periodicity and properties of the complex exponential:

$$\begin{aligned}2\pi c_n &= \int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}\,\mathrm{d}x\\ &= -\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}n(x-\frac{\pi}{n})}\,\mathrm{d}x\\ &= -\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(x+\tfrac{\pi}{n})\mathrm{e}^{-inx}\,\mathrm{d}x\\ &= -\int_{-\pi}^{\pi} f(x+\tfrac{\pi}{n})\mathrm{e}^{-inx}\,\mathrm{d}x,\end{aligned}$$

and so

$$c_n = \tfrac{1}{4\pi}\int_{-\pi}^{\pi}\big(f(x)-f(x+\tfrac{\pi}{n})\big)\mathrm{e}^{-inx}\,\mathrm{d}x$$

as required. If $\big(\rho_\varepsilon\big)_{\varepsilon>0}$ is the standard mollifier on $\mathbb{R}$ and $f_\varepsilon = \rho_\varepsilon * f$, then

$$\begin{aligned}4\pi|c_n| &\leq \int_{-\pi}^{\pi}\big|f(x)-f(x+\tfrac{\pi}{n})\big|\,\mathrm{d}x\\ &\leq \int_{-\pi}^{\pi}\big|f(x)-f_\varepsilon(x)\big|\,\mathrm{d}x\\ &\quad+\int_{-\pi}^{\pi}\big|f_\varepsilon(x)-f_\varepsilon(x+\tfrac{\pi}{n})\big|\,\mathrm{d}x\\ &\quad+\int_{-\pi}^{\pi}\big|f_\varepsilon(x+\tfrac{\pi}{n})-f(x+\tfrac{\pi}{n})\big|\,\mathrm{d}x\\ &= 2\int_{-\pi}^{\pi}\big|f(x)-f_\varepsilon(x)\big|\,\mathrm{d}x\\ &\quad+\int_{-\pi}^{\pi}\big|f_\varepsilon(x)-f_\varepsilon(x+\tfrac{\pi}{n})\big|\,\mathrm{d}x.\end{aligned}$$

The conclusion follows from this.
(b) Since $t_n \to 0$ as $n \to \infty$ given $k \in \mathbb{N}$ we can find $n_k \in \mathbb{N}$ so $|t_{n_k}| \leq 2^{-k}$. Proceeding inductively we can arrange that also $n_k < n_{k+1}$, and hence defining

$$f(x) = \sum_{k=1}^{\infty} t_{n_k} \mathrm{e}^{\mathrm{i}n_k x}$$

we see, using the Weierstrass M-test, that the series is uniformly convergent in $x \in \mathbb{R}$, and so f is continuous. The desired conclusion follows from this.
(c) This is easy since from (a) we have for $n \neq 0$:

$$\begin{aligned} 4\pi|c_n| \leq & \int_{-\pi}^{\pi} \left| f(x) - f(x + \tfrac{\pi}{n}) \right| \mathrm{d}x \\ \leq & \int_{-\pi}^{\pi} c \left| \frac{\pi}{n} \right|^{\alpha} \mathrm{d}x \\ = & 2c\pi^{1+\alpha}|n|^{-\alpha} \end{aligned}$$

as required.
(d) Let $c_n(f')$, $c_n(f)$ be the Fourier coefficients for f', f, respectively. Using similar notation for the mollified function $f_\varepsilon = \rho_\varepsilon * f$ and its derivative $f'_\varepsilon = \rho_\varepsilon * f'$ we find by partial integration:

$$\begin{aligned} c_n(f'_\varepsilon) = & \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f'_\varepsilon(x) \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x \\ = & \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f_\varepsilon(x) \mathrm{i}n \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x \\ = & c_n(f_\varepsilon)\mathrm{i}n. \end{aligned}$$

Because $c_n(f_\varepsilon) \to c_n(f)$ and $c_n(f'_\varepsilon) \to c_n(f')$ as $\varepsilon \searrow 0$ it follows that also

$$c_n(f) = \frac{c_n(f')}{\mathrm{i}n} \quad \text{for} \quad n \neq 0.$$

Because

$$|c_n(f)| \leq \tfrac{1}{2}\left(|c_n(f')|^2 + \tfrac{1}{n^2}\right) \quad \text{for} \quad n \neq 0,$$

and $\left(c_n(f')\right)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ by Plancherel's theorem for Fourier series it follows from the Weierstrass M-test that the Fourier series for f is absolutely and uniformly convergent.