



B4.2 Functional Analysis II

Consultation Session 1

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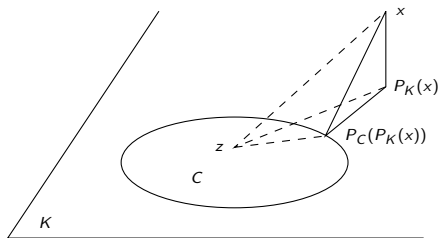
- (a) Projection theorem + Pythagoras' theorem.
- (b) Let H be a Hilbert space, $C \subset K \subset H$ be non-empty closed convex subsets and P_C and P_K be the projections to those convex sets. Show that, if K is a subspace, then $P_C = P_C \circ P_K$, but this need not hold otherwise.
- (c) Let $H = L^2(\mathbb{R})$ and

$$K = \{g \in H : g(t) = 0 \text{ for almost all } t \in (-\infty, 0)\},$$
$$C = \{g \in K : g(t) \geq 0 \text{ for almost all } t \in \mathbb{R}\},$$

Let f be a real-valued function in H . Find $P_K(f)$ and $P_C(f)$.

Paper 2013–Q2(b)

K is a closed subspace.



- Fix $x \in H$. We need to show that $P_C(P_K(x)) = P_C(x)$, i.e.

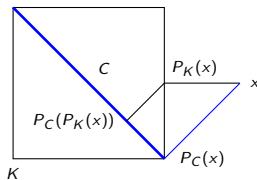
$$\|x - z\| \geq \|x - P_C(P_K(x))\| \quad (1)$$

for all $z \in C$.

- Let $y = P_K(x)$. By Pythagoras' theorem, $\|x - z\|^2 = \|x - y\|^2 + \|y - z\|^2$.
- By definition of P_C , $\|y - z\| \geq \|y - P_C(y)\|$.
- So, $\|x - z\|^2 = \|x - y\|^2 + \|y - z\|^2 \geq \|x - y\|^2 + \|y - P_C(y)\|^2 = \|x - P_C(y)\|^2$, where we have used Pythagoras' theorem once more time. This gives (1).

Example of K (not a subspace) and C for which $P_C \neq P_C \circ P_K$.
Many such examples e.g.

$$K = [0, 1]^2 \subset \mathbb{R}^2, \quad C = \{(a, b) : a + b = 1, 0 \leq a, b \leq 1\}.$$



Paper 2013–Q2(c)

Let $H = L^2(\mathbb{R})$ and

$$K = \{g \in H : g(t) = 0 \text{ for almost all } t \in (-\infty, 0)\},$$

$$C = \{g \in K : g(t) \geq 0 \text{ for almost all } t \in \mathbb{R}\},$$

Let f be a real-valued function in H . Find $P_K(f)$ and $P_C(f)$.

- H and K are clearly closed and convex. Hence P_C and P_K are well-defined.
- Let $g \in K$, then

$$\|f - g\|^2 = \int_{-\infty}^0 |f(t)|^2 dt + \int_0^{\infty} |f(t) - g(t)|^2 dt.$$

In order for this to be smallest, we need $g(t) = f(t)$ for all most all $t \geq 0$. This means

$$P_K(f) = f\chi_{[0,\infty)}.$$

...

$$C = \{g \in K : g(t) \geq 0 \text{ for almost all } t \in \mathbb{R}\},$$

... Find ... $P_C(f)$.

- Let $g \in C$, then

$$\|f - g\|^2 = \int_{-\infty}^0 |f(t)|^2 dt + \int_0^{\infty} |f(t) - g(t)|^2 dt$$

- If we minimize $|f(t) - g(t)|^2$ for each t under the constraint that $g(t) \geq 0$, we get $g(t) = f^+(t)$. This gives

$$\|f - g\|^2 \geq \int_{-\infty}^0 |f(t)|^2 dt + \int_0^{\infty} |f(t) - f^+(t)|^2 dt.$$

From here we see that

$$P_C(f) = f^+ \chi_{[0, \infty)}.$$

Paper 2013–Q3

Let H and K be Hilbert spaces, $T \in \mathcal{B}(H, K)$.

- (a) Definition of adjoint operator T^* . Show $\|T\| = \|T^*\| = \|T^*T\|^{1/2}$, $(T^*T)^* = T^*T$ and $\text{Ker } T = \text{Ker } T^*T$. Show for a projection P on H ,

$$P = P^* \Rightarrow \text{Ker } P = (\text{Im } P)^\perp \Rightarrow \|P\| \leq 1.$$

- (b) Definition of T being an isometry. Show that if this is the case then $\langle Tx, Tw \rangle = \langle x, w \rangle$ for all $x, w \in H$.
- (c) T is said to be a partial isometry if there exist closed subspaces H_1 and K_1 of H and K such that T maps H_1 isometrically into K_1 while $T = 0$ on H_1^\perp . Show that if T is a partial isometry, then so is T^* and that both T^*T and TT^* are projections. Conversely, show that if T^*T is a projection, then T is a partial isometry.

Paper 2013–Q3(a)(i)–(iii)

Let H and K be Hilbert spaces, $T \in \mathcal{B}(H, K)$. Define T^* . Show $\|T\| = \|T^*\| = \|T^*T\|^{1/2}$, $(T^*T)^* = T^*T$ and $\text{Ker } T = \text{Ker } T^*T$.

- I'll not go over bookwork. But I'd like to point out that the underlying identity that defines T^* is $\langle Tx, y \rangle = \langle x, T^*y \rangle$. The reason for the existence and uniqueness of T^* is a consequence of the Riesz representation theorem. It also gives $\|T\| = \|T^*\|$. In practice, one rearranges $\langle Tx, y \rangle$ as the inner product of x with something else, and read off T^* .

- For $x \in H$, we have

$$\|T^*Tx\| \leq \|T^*\|\|Tx\| \leq \|T^*\|\|T\|\|x\| = \|T\|^2\|x\| \text{ and so } \|T^*T\| \leq \|T\|^2.$$

Conversely

$$\|Tx\|^2 = \langle Tx, Tx \rangle = |\langle x, T^*Tx \rangle| \leq \|x\|\|T^*Tx\| \leq \|T^*T\|\|x\|^2.$$

This gives $\|Tx\| \leq \|T^*T\|^{1/2}\|x\|$ and so $\|T\| \leq \|T^*T\|^{1/2}$. We thus have $\|T\| = \|T^*T\|^{1/2}$.

Paper 2013–Q3(a)(i)–(iii)

Let H and K be Hilbert spaces, $T \in \mathcal{B}(H, K)$. Define T^* . Show $\|T\| = \|T^*\| = \|T^*T\|^{1/2}$, $(T^*T)^* = T^*T$ and $\text{Ker } T = \text{Ker } T^*T$.

- The proof of $(T^*T)^* = T^*T$ is routine.
- Let us now show that $\text{Ker } T = \text{Ker } T^*T$.

Clearly if $Tx = 0$ then $T^*Tx = 0$ and so $\text{Ker } T \subset \text{Ker } T^*T$.
Conversely, suppose $T^*Tx = 0$. Then

$$0 = \langle x, T^*Tx \rangle = \langle Tx, Tx \rangle = \|Tx\|^2,$$

which implies $Tx = 0$. This proves $\text{Ker } T \supset \text{Ker } T^*T$, whence $\text{Ker } T = \text{Ker } T^*T$.

Paper 2013–Q3(a)(iv)

Let H be a Hilbert space. Show for a projection P on H ,

$$P = P^* \Rightarrow \text{Ker } P = (\text{Im } P)^\perp \Rightarrow \|P\| \leq 1.$$

- Recall that $P \in \mathcal{B}(H)$ is a projection if $P^2 = P$.
- We know that for every bounded linear operator P ,

$$\text{Ker } P = (\text{Im } P^*)^\perp \quad \text{and} \quad \overline{\text{Im } P} = (\text{Ker } P^*)^\perp.$$

(Proof?)

The first stated deduction is clear from the above.

- Suppose now that $\text{Ker } P = (\text{Im } P)^\perp$ and we would like to show that $\|P\| \leq 1$, i.e. $\|Px\| \leq \|x\|$ for all $x \in H$.

Paper 2013–Q3(a)(iv)

Let H be a Hilbert space. Show for a projection P on H ,

$$P = P^* \Rightarrow \text{Ker } P = (\text{Im } P)^\perp \Rightarrow \|P\| \leq 1.$$

- How may we use $\text{Ker } P = (\text{Im } P)^\perp$? By the projection theorem, this implies that every $x \in H$ is written uniquely as

$$x = y + z \text{ where } Py = 0 \text{ and } z \in \overline{\text{Im } P}.$$

(Note the closure.)

- Claim: $\text{Im } P$ is closed. To see this observe that $P^2 = P$ implies that $\text{Im } P = \text{Ker } (I - P)$ which is closed.
- Hence every $x \in H$ is written uniquely as

$$x = y + z \text{ where } Py = 0 \text{ and } z \in \text{Im } P.$$

Applying P to both side, we have $Px = Pz = z$ (since $z = Pw$ for some w and $P^2 = P$).

Paper 2013–Q3(a)(iv)

Let H be a Hilbert space. Show for a projection P on H ,

$$P = P^* \Rightarrow \text{Ker } P = (\text{Im } P)^\perp \Rightarrow \|P\| \leq 1.$$

- Putting things together, we see that every x is written as

$$x = y + z = y + Px \text{ where } Py = 0$$

where $y \perp z = Px$.

- By Pythagoras' theorem, we have

$$\|Px\|^2 \leq \|Px\|^2 + \|y\|^2 = \|x\|^2,$$

i.e. $\|P\| \leq 1$.

(Note we have actually proved that P is the orthogonal projection onto $\text{Im } P$ (which is closed).)

Paper 2013–Q3(b)

Let H and K be Hilbert spaces, $T \in \mathcal{B}(H, K)$.

Show that if T is an isometry then $\langle Tx, Tw \rangle = \langle x, w \rangle$ for all $x, w \in H$.

- This uses a standard polarisation argument.
- We start with $\|T(x + w)\|^2 = \|x + w\|^2$, $\|Tx\| = \|x\|$ and $\|Tw\| = \|w\|$. Expanding gives

$$\langle Tx, Tw \rangle + \langle Tw, Tx \rangle = \langle x, w \rangle + \langle w, x \rangle,$$

which means $\operatorname{Re} \langle Tx, Tw \rangle = \operatorname{Re} \langle x, w \rangle$.

- If the field is real, we are done. If the field is complex, we apply the above to ix and w to get that $\operatorname{Re} \langle T(ix), Tw \rangle = \operatorname{Re} \langle ix, w \rangle$. This means $\operatorname{Im} \langle Tx, Tw \rangle = \operatorname{Im} \langle x, w \rangle$ and so we are done.

Paper 2013–Q3(b)(i)

Let H and K be Hilbert spaces, $T \in \mathcal{B}(H, K)$.

Show that if T is a partial isometry, then so is T^* and that both T^*T and TT^* are projections.

- By definition, there exist closed subspaces $H_1 \subset H$ and $K_1 \subset K$ such that $T|_{H_1} : H_1 \rightarrow K_1$ is a surjective isometry and $T|_{H_1^\perp} = 0$.
- Claim $T^*|_{K_1} : K_1 \rightarrow H_1$ is a surjective isometry and $T^*|_{K_1^\perp} = 0$. This gives that T^* is a partial isometry.
- Indeed, we have $\text{Ker } T^* = (\text{Im } T)^\perp \stackrel{\text{why?}}{=} K_1^\perp$. This implies that $T^*|_{K_1^\perp} = 0$.
- We also have $\overline{\text{Im } T^*} = (\text{Ker } T)^\perp \stackrel{\text{why?}}{=} (H_1^\perp)^\perp = H_1$ (as H_1 is closed). In particular $\text{Im } (T^*|_{K_1}) = \text{Im } T^*$ is a dense subset of H_1 .

Paper 2013–Q3(b)(i)

Let H and K be Hilbert spaces, $T \in \mathcal{B}(H, K)$.

Show that if T is a partial isometry, then so is T^* and that both T^*T and TT^* are projections.

- Now suppose $k \in K_1$, we have $k = Th$ for some $h \in H_1$ and

$$\|T^*k\|^2 = \langle T^*k, T^*k \rangle = \langle TT^*k, k \rangle = \langle TT^*k, Th \rangle$$

using that T is isometric, we can continue this identity:

$$= \langle T^*k, h \rangle = \langle k, Th \rangle = \langle k, k \rangle = \|k\|^2.$$

So $T^*|_{K_1} : K_1 \rightarrow H_1$ is isometric.

- To see that $\text{Im}(T^*|_{K_1}) = H_1$, we take $h \in H_1$ and aim to show that $h = T^*k$ for some $k \in K_1$. Let $(h_n) \subset \text{Im}(T^*|_{K_1})$ be such that $h_n \rightarrow h$ (note $\text{Im}(T^*|_{K_1})$ is dense in H_1). Write $h_n = T^*k_n$ with $k_n \in K_1$, then $\|k_n - k_m\| = \|h_n - h_m\| \rightarrow 0$ and so (k_n) is Cauchy, hence convergent to some $k \in K_1$ (since K_1 is closed). By continuity $h = T^*k$.

Paper 2013–Q3(b)(i)

Let H and K be Hilbert spaces, $T \in \mathcal{B}(H, K)$.

Show that if T is a partial isometry, then so is T^* and that both T^*T and TT^* are projections.

- For the last statement, it suffices to show T^*T is a projection. The other part is obtained by swapping the role of T and T^* .
- Take $x, y \in H$. We have $\langle T^*TT^*Tx, y \rangle = \langle TT^*Tx, Ty \rangle$. Write $y = a + b$ where $a \in H_1$ and $b \in H_1^\perp$ so that $Ty = Ta$. Then

$$\begin{aligned}\langle T^*TT^*Tx, y \rangle &= \langle \underbrace{T T^*Tx}_{\in H_1}, \underbrace{T a}_{\in H_1} \rangle = \langle T^*Tx, a \rangle \\ &= \langle \underbrace{T^*Tx}_{\in H_1}, a + \underbrace{b}_{\in H_1^\perp} \rangle = \langle T^*Tx, y \rangle.\end{aligned}$$

Since x, y are arbitrary, this means $(T^*T)^2 = T^*T$ and so T^*T is a projection.

Paper 2013–Q3(b)(ii)

Let H and K be Hilbert spaces, $T \in \mathcal{B}(H, K)$.

Conversely, show that if T^*T is a projection then T is a partial isometry.

- We know that T^*T is self-adjoint. By (a), $\text{Ker } T = \text{Ker } T^*T$ and $\text{Ker } T^*T = (\text{Im } T^*T)^\perp$. In fact, we also know that $\text{Im } T^*T$ is closed and T^*T is the orthogonal projection on to $\text{Im } T^*T$.
- Let $H_1 = \text{Im } T^*T$ so that $H_1^\perp = \text{Ker } T$ (hence $T|_{H_1^\perp} = 0$).
- Claim: $T|_{H_1}$ is isometric. Let $h \in H_1$. Since T^*T is the orthogonal projection onto H_1 , $h = T^*Th$. Hence

$$\langle Th, Th \rangle = \langle T^*Th, h \rangle = \langle h, h \rangle = \|h\|^2,$$

i.e. T is isometric.

- Finally, let $K_1 = \overline{\text{Im } T|_{H_1}}$. Claim: $\text{Im } (T|_{H_1})$ is actually K_1 . This can be done as in the proof of $\text{Im } (T^*|_{K_1}) = H_1$ which we did earlier on in (i).

Let X be a complex Hilbert space, $T \in \mathcal{B}(X)$.

- Ⓒ Prove that if there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in X$, then T is injective and $\operatorname{Im} T$ is closed. Prove further that if T is self-adjoint, then T is invertible in $\mathcal{B}(X)$.
- Ⓓ Suppose that T is self-adjoint. Prove that $iI + T$ has an inverse and that $(iI + T)^{-1}(iI - T)$ is unitary.

Let X be a complex Hilbert space, $T \in \mathcal{B}(X)$.

Prove that if there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in X$, then T is injective and $\text{Im } T$ is closed. Prove further that if T is self-adjoint, then T is invertible in $\mathcal{B}(X)$.

- The injectivity is clear. We have seen the part about the closedness of $\text{Im } T$ when T is isometric. The proof now is the same.
- Suppose $(y_n = Tx_n) \subset \text{Im } T$ and $y_n \rightarrow y$ in X .
- Then $\|x_n - x_m\| \leq \delta^{-1}\|y_n - y_m\|$. So (x_n) is Cauchy, hence converges to some $x \in X$. Continuity gives $y = Tx$. This proves $\text{Im } T$ is closed.

Let X be a complex Hilbert space, $T \in \mathcal{B}(X)$.

Prove that if there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in X$, then T is injective and $\text{Im } T$ is closed. Prove further that if T is self-adjoint, then T is invertible in $\mathcal{B}(X)$.

- Suppose now T is self-adjoint. Then $0 = \text{Ker } T = \text{Ker } T^* = (\text{Im } T)^\perp$ and so $\text{Im } T$ is dense in X . Since $\text{Im } T$ is closed, we have that $\text{Im } T = X$, i.e. T is surjective.
- It follows that T is a bijection and has an inverse T^{-1} . It is clear that T^{-1} is linear. One also has $\|T^{-1}y\| \leq \delta^{-1}\|Ty\| = \delta^{-1}\|y\|$ and so T^{-1} is bounded.
Comment: There is no need to use inverse mapping theorem.

Let X be a complex Hilbert space, $T \in \mathcal{B}(X)$.

Suppose that T is self-adjoint. Prove that $iI + T$ has an inverse and that $(iI + T)^{-1}(iI - T)$ is unitary.

- We attempt to use (c). We compute

$$\|(iI + T)x\|^2 = \|x\|^2 + \|Tx\|^2 - 2i \operatorname{Im} \langle x, Tx \rangle.$$

- Since T is self-adjoint, $\langle x, Tx \rangle = \langle Tx, x \rangle = \overline{\langle x, Tx \rangle}$ and so $\langle x, Tx \rangle$ is real. Thus

$$\|(iI + T)x\|^2 = \|x\|^2 + \|Tx\|^2 \geq \|x\|^2.$$

- By (c), $iI + T$ is invertible.

Paper 2014–Q1(d)

Let X be a complex Hilbert space, $T \in \mathcal{B}(X)$.

Suppose that T is self-adjoint. Prove that $iI + T$ has an inverse and that $(iI + T)^{-1}(iI - T)$ is unitary.

- Let $U = (iI + T)^{-1}(iI - T)$. Applying what we just proved to $-T$, we have also that $iI - T$ is invertible and hence U is invertible.
- To conclude, we show that U is isometric by showing $U^*U = I$. First,

$$U^* = (-iI - T^*)(-iI + T^*)^{-1} = (iI + T)(iI - T)^{-1}.$$

- Next, since $iI + T$ and $iI - T$ commute, so do their inverses. It follows that

$$\begin{aligned} U^*U &= (iI + T)(iI - T)^{-1}(iI + T)^{-1}(iI - T) \\ &= (iI + T)(iI + T)^{-1}(iI - T)^{-1}(iI - T) = I. \end{aligned}$$