B4.2 Functional Analysis II
Consultation Session 2

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Paper 2014–Q2(c)

Let X be a Banach space.

1. (i) Bookwork: Boundedness of projection map.
2. (ii) For $n \geq 1$, let Y_n and Z_n be closed subspaces of X such that

$$Y_n \subseteq Y_{n+1}, \quad Z_n \supseteq Z_{n+1}, \quad X = Y_n \oplus Z_n.$$

   Let P_n be given by $P_n(y_n + z_n) = y_n$ for $y_n \in Y_n$ and $z_n \in Z_N$. Assume that for each $x \in X$, the limit $\lim_{n\to\infty} P_n x$ exists and denote this limit by Px. Prove that P is a bounded projection and that

$$\mathrm{Im}\, P = \overline{\bigcup_{n\geq 1} Y_n} \quad \text{and} \quad \mathrm{Ker}\, P = \bigcap_{n\geq 1} Z_n.$$

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- It is straightforward to check that P is linear. Also since $P_n^2 = P_n$, we have $P^2 = P$.
- For each x, $\{P_n x\}$ is bounded (since it's convergent), we have by the principle of uniform boundedness that $\{P_n\}$ is bounded in $\mathscr{B}(X)$, i.e. there exists $M \geq 0$ such that $\|P_n\| \leq M$ for all n¿
- It follows that

$$\|Px\| \leq \underbrace{\|(P_n x - Px)\|}_{\to 0} + \underbrace{\|P_n x\|}_{\leq M\|x\|} \quad \text{and so} \quad \|Px\| \leq M\|x\|.$$

  This means P is bounded.
- Now let $Y = \overline{\cup Y_n}$ and $Z = \cap Z_n$ which are closed subspaces of X. We need to show $\mathrm{Im}\, P = Y$ and $\mathrm{Ker}\, P = Z$.

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- Let us show that $\operatorname{Ker} P = Z$. For each $z \in Z$, we have $z \in Z_n$ for all n and so $P_n z = 0$ for all n. In particular, $Pz = \lim P_n z = 0$.
  Conversely, suppose $Pz = 0$. Then $z = \lim(I - P_n)z$. Note that $(I - P_n)z \in Z_n \subseteq Z_m$ if $n \geq m$. It follows that the sequence $((I - P_n)z)$ eventually belongs to Z_m for each m. Since Z_m are closed, we have that $z = \lim(I - P_n)z$ belongs to all Z_m, i.e. $z \in Z$.
- Next, we show $\operatorname{Im} P = Y$.
  If $y = Px = \lim P_n x$, then since $(P_n x) \subset \cup Y_n$ we have $y \in \operatorname{Im} P$. So $\operatorname{Im} P \subseteq Y$.
  Take $y \in \cup Y_n$ so that $y \in Y_m$ for some m. This implies that $y \in Y_n$ for all $n \geq m$. It follows that $P_n y = y$ for $n \geq m$ and so $Py = y$.
  By continuity, we have $Py = y$ for all $y \in Y$ and so $Y \subseteq \operatorname{Im} P$.

Paper 2015–Q2

Let X be a Hilbert space.

(a) Prove that a subset E of X is norm-bounded if for each $x \in X$ there exists a constant M_x such that

$$|\langle x, y \rangle| \leq M_x \text{ for all } y \in E.$$

(b) Let $T : X \to X$ be a bijective continuous linear operator. Prove that there is a constant $m > 0$ such that $y \in X$ and $\|T^* y\| = 1$ together imply $\|y\| \leq m$. Hence prove that T has a continuous inverse.

(c) Deduce that a surjective bounded linear operator from X to X maps open sets to open sets.

Paper 2015–Q2(a)

Let X be a Hilbert space. Prove that a subset E of X is norm-bounded if for each $x \in X$ there exists a constant M_x such that

$$|\langle x, y\rangle| \le M_x \text{ for all } y \in E. \tag{*}$$

- For $y \in X$, define a linear functional $\ell_y \in X^*$ by $\ell_y(x) = \langle x, y\rangle$. Note that $\|\ell_y\| = \|y\|$ in view of the Cauchy-Schwarz inequality (why?).
- Let $\mathscr{F} = \{\ell_y : y \in E\}$. Clearly E is bounded iff $\mathscr{F}$ is bounded in X^*.
  On the other hand, by the principle of uniform boundedness, $\mathscr{F}$ is bounded in X^* iff (*) holds. The conclusion follows.

Paper 2015–Q2(b)

Let X be a Hilbert space and $T : X \to X$ be a bijective continuous linear operator. Prove that there is a constant $m > 0$ such that $y \in X$ and $\|T^* y\| = 1$ together imply $\|y\| \leq m$. Hence prove that T has a continuous inverse.

- Let $E = \{y : \|T^* y\| = 1\}$. We need to show that E is bounded.
- By (a), we need to show that for every $x \in X$, the set $\{\langle x, y\rangle : y \in E\}$ is bounded.
- Fix $x \in X$. To make $T^* y$ shows up, we write $x = Tz$ which is possible as T is bijective. Then

  $$|\langle x, y\rangle| = |\langle Tz, y\rangle| = |\langle z, T^* y\rangle$$

  which by the Cauchy-Schwarz inequality is bounded from above by

  $$\leq \|z\| \|T^* y\| = \|z\|.$$

  We conclude from (a) that E is bounded.

Paper 2015–Q2(b)

Let X be a Hilbert space and $T : X \to X$ be a bijective continuous linear operator. Prove that there is a constant $m > 0$ such that $y \in X$ and $\|T^*y\| = 1$ together imply $\|y\| \leq m$. Hence prove that T has a continuous inverse.

- From the above, we have that, for every $y \in X$, $\|T^*(y/\|T^*y\|)\| = 1$ and so $\|y\|/\|T^*y\| \leq m$ i.e.

$$\|T^*y\| \geq m^{-1}\|y\| \text{ for all } y \in X. \tag{**}$$

- We knew that this implies $\mathrm{Im}\, T^*$ is closed. Since $(\mathrm{Im}\, T^*)^\perp = \mathrm{Ker}\, T = 0$, we have that $\mathrm{Im}\, T^* = X$. Also $\mathrm{Ker}\, T^* = (\mathrm{Im}\, T)^\perp = 0$. So T^* is invertible with bounded inverse (in view of (**)).
- Properties of adjoints imply that T has bounded inverse.

Paper 2015–Q2(c)

Let X be a Hilbert space and $T : X \to X$ be a surjective continuous linear operator. Prove that T maps open sets to open sets.

- We know (from lectures) that T is open if there exists $\delta_0 > 0$ such that
$$T(B(0,1)) \supseteq B(0,\delta_0).$$
- Let $Y = \mathrm{Ker}\, T$ and $Z = Y^\perp$. Both of these are Hilbert subspaces of X.
- Let $S = T|_Z : Z \to X$ which is a bijective bounded linear operator. Though the domain and target spaces are different, the same proof of (b) gives that S has a bounded inverse.
- This means that there exists $\delta > 0$ so that $\|Tz\| = \|Sz\| \geq \delta\|z\|$ for all $z \in Z$.
- Now if $\|x\| < \delta$ and $x = Tz$, then $\|z\| \leq \delta^{-1}\|x\| = 1$, i.e. $B(0,\delta) \subseteq T(B(0,1))$.

Paper 2017–Q2

(a) (ii) Let $X = C([0,1])$. Define $A_n : X \to X$ so that $(A_n x)(t) = x(t^{1+\frac{1}{n}})$. Show that A_n converges strongly to the identity operator on X (i.e. $A_n x \to x$ for all $x \in X$.)

(b) (i) Let X be a Banach space, Y be a normed space, $T \in \mathscr{B}(X, Y)$. Assume $\overline{T(B_X(0,1))} \supseteq B_Y(0,\varepsilon)$ for some $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that $T(B_X(0,1)) \supseteq B_Y(0,\delta)$.

(iii) Let X and Y be Banach spaces and $T \in \mathscr{B}(X, Y)$. Prove that if $T(X)$ is not closed in Y, then $T(X)$ is a countable union of nowhere dense subsets of Y.

(iv) Show that $C([0,1])$ is a countable union of nowhere dense subsets of $L^2(0,1)$.

Paper 2017–Q2(a)(ii)

Let $X = C([0,1])$. Define $A_n : X \to X$ so that $(A_n x)(t) = x(t^{1+\frac{1}{n}})$. Show that A_n converges strongly to the identity operator on X (i.e. $A_n x \to x$ for all $x \in X$.)

- Fix $x \in X$. We need to show

$$\sup_{t\in[0,1]} |x(t^{1+\frac{1}{n}}) - x(t)| \to 0 \text{ as } n \to \infty.$$

- By uniform continuity of x, it suffices to show

$$\sup_{t\in[0,1]} |t^{1+\frac{1}{n}} - t| \to 0 \text{ as } n \to \infty.$$

- Fix some small $\epsilon > 0$.
  + If $t \le \epsilon$, then $|t^{1+\frac{1}{n}} - t| < \epsilon$ for all n.
  + If $t > \epsilon$, then $|t^{1+\frac{1}{n}} - t| \le |t^{\frac{1}{n}} - 1| \le 1 - \epsilon^{\frac{1}{n}} < \epsilon$ for all large n.

Paper 2017–Q2(b)(i)

Let X be a Banach space, Y be a normed space, $T \in \mathscr{B}(X, Y)$. Assume $\overline{T(B_X(0,1))} \supseteq B_Y(0,\varepsilon)$ for some $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that $T(B_X(0,1)) \supseteq B_Y(0,\delta)$.

- We prove the statement with $\delta = \varepsilon/2$.
- As $\overline{T(B_X(0,1))} \supset B_Y(0,\varepsilon)$, we have $\overline{T(B_X(0,r))} \supset B_Y(0,\varepsilon r)$.
- Take $y \in B_Y(0,\varepsilon/2) \subset \overline{T(B_X(0,1/2))}$.
- Take $x_1 \in B_X(0,1/2)$ such that $\|y - Tx_1\| < \varepsilon/4$. Then

$$y - Tx_1 \in B_Y(0,\varepsilon/4) \subset \overline{T(B_X(0,1/4))}.$$

- Take $x_2 \in B_X(0,1/4)$ such that $\|(y - Tx_1) - Tx_2\| < \varepsilon/8$.

Paper 2017–Q2(b)(i)

- Inductively, we obtain $x_k \in B_X(0, 2^{-k})$ such that

$$\|y - T(x_1 + \ldots + x_k)\| < \varepsilon\, 2^{-k-1}.$$

- Easy to check: The series $\sum x_k$ converges to some s satisfying $y = Ts$ and

$$\|s\| < \sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 2^{-k} = 1, \text{ i.e. } s \in B_X(0,1).$$

We have thus shown that $B_Y(0, \varepsilon/2) \subset T(B_X(0,1))$.

Paper 2017–Q2(b)(iii)

Let X and Y be Banach spaces and $T \in \mathscr{B}(X, Y)$. Prove that if $T(X)$ is not closed in Y, then $T(X)$ is a countable union of nowhere dense subsets of Y.

- As $TX = \cup_n T(B_X(0, n))$, it suffices to show that $T(B_X(0, 1))$ is nowhere dense.
- Suppose by contradiction that $\overline{T(B_X(0, 1))}$ has non-empty interior, i.e. $\overline{T(B_X(0, 1))} \supset B_Y(y_0, r_0)$ for some $r_0 > 0$.
- Then we also have that $\overline{T(B(0, 1))} \supset B_Y(-y_0, r_0)$, which in turns implies that

$$\overline{T(B_X(0, 1))} \supset B_Y(0, r_0) = \frac{1}{2}(B_Y(y_0, r_0) + B_Y(-y_0, r_0)).$$

- By (i), we then have $T(B_X(0, 1)) \supset B_Y(0, \delta)$ for some $\delta > 0$.
- This implies that $T(B_X(0, n)) \supset B_Y(0, \delta n)$ and so $TX = Y$, contradicting the fact that TX is not closed.

Paper 2017–Q2(b)(iv)

Show that $C([0,1])$ is a countable union of nowhere dense subsets of $L^2(0,1)$.

- Let $X = C[0,1]$ and $Y = L^2(0,1)$ and equip them with their standard norms to make them Banach spaces.
- Let $T : X \to Y$ be the natural injection, which is clearly bounded linear.
- Clearly T is not surjective and so, by (iii), $X = TX$ is a countable union of nowhere dense sets in $L^2(0,1)$.