B4.2 Functional Analysis II
Consultation Session 3

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Paper 2014 – Q3

(a) Bookwork.

(b) Let D denote the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in $\mathbb{C}$ and consider $L^2(D)$ with area measure. Let $A^2(D)$ be the set of functions $f : D \to \mathbb{C}$ such that f is holomorphic and $|f|^2$ is integrable. We identify with a subspace of $L^2(D)$. You are given that $A^2(D)$ is closed in $L^2(D)$.

Let $e_n = \left(\frac{n+1}{\pi}\right)^{1/2} z^n$, $n = 0, 1, \ldots$

  (i) Prove that $(e_n)_{n \geq 0}$ is a complete orthonormal sequence in $A^2(D)$.

  (ii) Prove that if $\sum |a_k|^2$ converges then the function $\sum_{k=0}^{\infty} (k+1)^{1/2} a_k z^k$ is holomorphic in D. Is the converse true?

Paper 2014 – Q3(b)(i)

Prove that $e_n = \left(\frac{n+1}{\pi}\right)^{1/2} z^n$, $n = 0, 1, \ldots$ form a complete orthonormal sequence in $A^2(D)$.

- It's straightforward to check that (e_n) is an orthonormal sequence in $A^2(D)$.
- To show that it's complete, suppose that $f \in A^2(D)$ with $\langle f, e_n \rangle = 0$ for all e_n, and we need to show that $f \equiv 0$.
- We know that f has a Taylor series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

which converges uniformly on any disk $D(0, R) \subset D$ with $R < 1$ and $f \equiv 0$ if and only if $a_k = 0$ for all k.

Paper 2014 – Q3(b)(i)

- Let us compute $\langle f, e_n \rangle$.
- Since $f \in L^2(D)$, we have by the dominated convergence theorem that $f - f|_{D(0,R)} \to 0$ as $R \to 1$. It follows that

$$\langle f, e_n \rangle = \left(\frac{n+1}{\pi}\right)^{1/2} \lim_{R\to 1} \int_{D(0,R)} f(z) \bar{z}^n \, dA.$$

Using the uniform convergence of the Taylor series we then have

$$\begin{aligned} \langle f, e_n \rangle &= \left(\frac{n+1}{\pi}\right)^{1/2} \lim_{R\to 1} \sum_{k=0}^{\infty} \int_{D(0,R)} a_k z^k \bar{z}^n \, dA \\ &= \left(\frac{n+1}{\pi}\right)^{1/2} \lim_{R\to 1} \sum_{k=0}^{\infty} \int_0^R \int_0^{2\pi} a_k r^{k+n+1} e^{i(k-n)\theta} \, d\theta \, dr \\ &= \left(\frac{n+1}{\pi}\right)^{1/2} \lim_{R\to 1} \frac{\pi}{n+1} a_n R^{2n+2} = \left(\frac{\pi}{n+1}\right)^{1/2} a_n. \end{aligned}$$

- We deduce that $a_n = 0$ for all n and so $f \equiv 0$.

Paper 2014 – Q3(b)(i)

Prove that if $\sum |a_k|^2$ converges then the function $\sum_{k=0}^{\infty} (k+1)^{1/2} a_k z^k$ is holomorphic in D. Is the converse true?

- We have $\sum_{k=0}^{\infty} (k+1)^{1/2} a_k z^k = \frac{1}{\sqrt{\pi}} \sum a_k e_k$. Now if $\sum |a_k|^2$ converges, then $\sum a_k e_k$ belongs to the closed linear span of (e_n), i.e. $A^2(D)$, and hence is holomorphic.
- The converse doesn't hold: If f is holomorphic in D, it is not necessary that f can be written in the form $f = \sum_{k=0}^{\infty} (k+1)^{1/2} a_k z^k$ with $\sum |a_k|^2 < \infty$, as this latter means that $f \in A^2(D)$.
  To confirm this, we only need to exhibit a function f which is holomorphic in D but is not square integrable in D. We can take for example $f(z) = (1-z)^{-1}$. (This corresponds to $a_k = (k+1)^{-1/2}$ which is clearly not square summable.)

Paper 2017 – Q3

(a) Bookwork.

(b) Let X be a Hilbert space, $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis in X and consider

$$S_2(X) = \left\{ T : X \to X \middle| T \text{ is linear }, \|T\|_2^2 := \sum_k \|Te_k\|^2 < \infty \right\}.$$

(i) Show that $\{h_k\}_{k=1}^{\infty}$ is an orthonormal basis in X, then $\sum_k \|Te_k\|^2 = \sum_k \|T^* h_k\|^2$. Deduce that $\|T\|_2$ is independent of choice of orthonormal basis in X.

(ii) Prove that $\|T\| \leq \|T\|_2$ for any $T \in S_2(X)$.

(iii) Show that $S_2(X)$ is a Hilbert space with inner product $\langle S, T\rangle_{S_2(X)} = \sum_k \langle Se_k, Te_k\rangle$.

(c) Let $X = \ell^2(\mathbb{C})$. For which $\lambda = (\lambda_k)$ does the operator $T((x_k)) = (\lambda_k x_k)$ belong to $S_2(X)$?

Paper 2017 – Q3(b)(i)

Let $T : X \to X$ be linear. Prove that if $\{e_k\}$ and $\{h_k\}$ are ONB for X, then $\sum_k \|Te_k\|^2 = \sum_k \|T^* h_k\|^2$. Deduce that $\|T\|_2$ is independent of ONB.

- We have by Parseval's identity that

$$\|Te_k\|^2 = \sum_j |\langle Te_k, h_j\rangle|^2 = \sum_j |\langle e_k, T^* h_j\rangle|^2.$$

- Summing over k and interchanging order of summation we get

$$\sum_k \|Te_k\|^2 = \sum_k \sum_j |\langle e_k, T^* h_j\rangle|^2 = \sum_j \sum_k |\langle e_k, T^* h_j\rangle|^2.$$

- Using Parseval's identity again for the inner sum we get

$$\sum_k \|Te_k\|^2 = \sum_j \|T^* h_j\|^2, \text{ as wanted.}$$

- The independence of $\|T\|_2$ on the choice of ONB is then clear.

Paper 2017 – Q3(b)(ii)

Let $T \in S_2(X)$. Show that $\|T\| \leq \|T\|_2$.

- Since $\|T\| = \|T^*\|$, we show instead that $\|T^*\| \leq \|T\|_2$. We fix any $x \in X$ and show that $\|T_* x\| \leq \|T\|_2 \|x\|$.
- We have by Parseval's identity that

$$\|T^* x\|^2 = \sum_k |\langle T^* x, e_k\rangle|^2 = \sum_k |\langle x, Te_k\rangle|^2.$$

- Using Cauchy-Schwarz' inequality, we are led to

$$\|T^* x\|^2 \leq \|x\|^2 \sum_k \|Te_k\|^2 = \|x\|^2 \|T\|_2^2,$$

which gives what we want.

Paper 2017 – Q3(b)(iii)

Show that $S_2(X)$ is a Hilbert space with inner product $\langle S, T\rangle_{S_2(X)} = \sum_k \langle Se_k, Te_k\rangle$.

- I'll leave it to you to verify that $\langle\cdot,\cdot\rangle_{S_2(X)}$ is an inner product.
- It remains to prove completeness.
- We suppose that $(T_n) \subset S_2(X)$ is Cauchy, i.e. $\|T_n - T_m\|_2 \stackrel{n,m\to\infty}{\longrightarrow} 0$. We'd like to show that there exists $T \in S_2(X)$ such that $\|T_n - T\|_2 \stackrel{n\to\infty}{\longrightarrow} 0$.
- Since $\|\cdot\| \leq \|\cdot\|_2$, the fact that (T_n) is Cauchy with respect to $\|\cdot\|_2$ implies that (T_n) is Cauchy with respect to $\|\cdot\|$. Hence T_n converges in $(\mathscr{B}(X), \|\cdot\|)$ to some $T \in \mathscr{B}(X)$.

Paper 2017 – Q3(b)(iii)

- Let us check that $T \in S_2(X)$. Indeed, by Fatou's lemma (discrete form – see Integration),

$$\|T\|_2 = \sum_k \|Te_k\|^2 \leq \liminf_{n\to\infty} \sum_k \|T_n e_k\|^2 \leq \sup \|T_n\|_2^2 < \infty,$$

  and so $T \in S_2(X)$.
- We next show that $\|T_n - T\|_2 \to 0$.
  - ⋆ Fix some $\epsilon > 0$. For large N, we have

$$\sum_k \|(T_n - T_m)e_k\|^2 = \|T_n - T_m\|_2^2 \leq \epsilon \text{ for } m, n > N.$$

  - ⋆ Sending $m \to \infty$ and using Fatou's lemma we get

$$\sum_k \|(T_n - T)e_k\|^2 \leq \liminf_{m\to\infty} \sum_k \|(T_n - T_m)e_k\|^2 \leq \epsilon \text{ for } n > N.$$

    But this means $\|T_n - T\|_2^2 \leq \epsilon$ for $n > N$. So $\|T_n - T\|_2 \to 0$.

Paper 2017 – Q3(c)

Let $X = \ell^2(\mathbb{C})$. For which $\lambda = (\lambda_k)$ does the operator $T((x_k)) = (\lambda_k x_k)$ belong to $S_2(X)$?

- Take the standard orthonormal basis $\{e_k\}$ of X. Then

$$\|T\|_2^2 = \sum_k |\lambda_k|^2.$$

- So $T \in S_2(X)$ if and only if $\lambda \in \ell^2(\mathbb{C})$. (Note that if $\lambda \in \ell^2(\mathbb{C})$ then $\lambda \in \ell^\infty$ and so T maps X into X.)

Paper 2012–Q3

Parts (a)-(c) are bookwork. Part (d)(i)-(iii) was covered in the 2017 paper. We will only deal with (d)(iv).

Suppose $T \in S_2(X)$. For $r \geq 1$, define a bounded linear operator $T_r : X \to X$ by

$$T_r x = \sum_{n=1}^{r} \langle x, e_n \rangle T e_n.$$

Prove that $\lim_{r\to\infty} \|T - T_r\| = 0$.

- Note that $(T - T_r)e_n = \begin{cases} 0 & \text{if } n \leq r, \\ Te_n & \text{if } n > r. \end{cases}$
- Thus

$$\|T - T_r\|_2^2 = \sum_{n=r+1}^{\infty} \|Te_n\|^2 \to 0 \text{ as } r \to \infty.$$

- Since $\|T - T_r\| \leq \|T - T_r\|_2$, we deduce that $\lim_{r\to\infty} \|T - T_r\| = 0$.