# B4.2 Functional Analysis II
# Consultation Session 4

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# Paper 2012 – Q1

The problem is not directly related to spectral theory and could have been included in Session 1. However, it's a good warm up exercise for spectral theory.

... Some bookwork part ...

Let $X$ be a Hilbert space, $S : X \to X$ be a bounded linear operator with $\|S\| = 1$, and let

$$T_N = \frac{1}{n} \sum_{r=0}^{n-1} S^r.$$

Prove that

1. $T_n x \to x$ for $x \in \mathrm{Ker}\,(I - S)$;
2. $T_n x \to 0$ for $x \in \mathrm{Im}\,(I - S)$;
3. $\mathrm{Im}\,(I - S)^\perp = \mathrm{Ker}\,(I - S^*)$;
4. $\mathrm{Ker}\,(I - S^*) = \mathrm{Ker}\,(I - S)$.

Deduce that $T_n$ converges strongly to an operator $T$ which you should identify.

# Paper 2012 – Q1

- Part (iii) is bookwork and we saw in the course that this has some importance in the discussion of spectral theory on Hilbert spaces.
- Let us start with (i). If $x \in \mathrm{Ker}\,(I - S)$, then $x = Sx$ and so $T_n x = x$.
- Consider now (ii). If $x \in \mathrm{Im}\,(I - S)$, then $x = y - Sy$ for some $y$. It follows that

$$\begin{aligned} T_n x &= \frac{1}{n}(x + Sx + \ldots + S^{n-1}x) \\ &= \frac{1}{n}(y - Sy + Sy - S^2 y + \ldots + S^{n-1}y - S^n y) \\ &= \frac{1}{n}(y - S^n y). \end{aligned}$$

Since $\|y - S^n y\| \leq \|y\| + \|S^n\|\|y\| \leq \|y\| + \|S\|^n\|y\| = 2\|y\|$, we then have that $T_n x \to 0$.

# Paper 2012 – Q1

- Now consider (iv). We need to show $\mathrm{Ker}\,(I - S^*) = \mathrm{Ker}\,(I - S)$.
- We have to use the fact that $\|S\| = 1$. (Without this, in general, $\mathrm{Ker}\,(I - S^*) \neq \mathrm{Ker}\,(I - S)$.)
- It suffices to show $\mathrm{Ker}\,(I - S^*) \subseteq \mathrm{Ker}\,(I - S)$. The converse follow by applying this to $S^*$.
- Indeed, take $x \in \mathrm{Ker}\,(I - S^*)$ so that $x = S^* x$. Then

  $$\|x\|^2 = |\langle x, S^* x\rangle| = |\langle Sx, x\rangle| \leq \|Sx\|\|x\| \leq \|S\|\|x\|^2.$$

  Since $\|S\| = 1$, this implies that by the equality case of Cauchy-Schwarz' inequality that $Sx = \lambda x$ with $|\lambda| = 1$. Noting also that the above gives $\langle Sx, x\rangle = \|x\|^2$ which is real and non-negative. We thus have that $Sx = x$, i.e. $x \in \mathrm{Ker}(I - S)$.
- The last part follows from the early bookwork part with $Y = \mathrm{Ker}(I - S) = \mathrm{Ker}(I - S^*)$, $Z = \mathrm{Ker}(I - S^*) + \mathrm{Im}(I - S)$, and $T$ being the orthogonal projection operator onto $\mathrm{Ker}(I - S)$.

# Paper 2016 – Q3

Let $X \neq 0$ be a complex Hilbert space and $T \in \mathscr{B}(X)$.

(a) Bookwork.

(b) Let $V(T) = \{\lambda \in \mathbb{C} : \exists x \in X \text{ with } \|x\| = 1 \text{ and } \langle Tx, x\rangle = \lambda\}$.

  (i) Show that $\lambda \in V(T)$ implies $|\lambda| \leq \|T\|$.
  (ii) Show that if $T$ is self-adjoint, then $V(T) \subset \mathbb{R}$.

(c) Suppose now $T$ is self-adjoint, $M = \sup V(T)$ and $m = \inf V(T)$.

  (i) Show that for any $w \in X$, $m\|w\|^2 \leq \langle Tw, w\rangle \leq M\|w\|^2$.
  (ii) ... Show that $M$ belongs to the approximate point spectrum of $T$.
  (iii) ... Express $\|T\|$ in terms of $m$ and $M$. Deduce that either $\|T\|$ or $-\|T\|$ belongs to $\sigma(T)$.

# Paper 2016 – Q3(b)

Let $X \neq 0$ be a complex Hilbert space and $T \in \mathscr{B}(X)$.
Let $V(T) = \{\lambda \in \mathbb{C} : \exists x \in X \text{ with } \|x\| = 1 \text{ and } \langle Tx, x\rangle = \lambda\}$.

1. Show that $\lambda \in V(T)$ implies $|\lambda| \leq \|T\|$.
2. Show that if $T$ is self-adjoint, then $V(T) \subset \mathbb{R}$.

This whole part is straightforward.

- (i) is follows from Cauchy-Schwarz' inequality: For a suitable $x$,

$$|\lambda| = |\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|.$$

- For (ii), if $T$ is self-adjoint and $\lambda \in V(T)$, then for a suitable $x$,

$$\lambda = \langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} = \bar{\lambda},$$

  and so $\lambda$ is real.

# Paper 2016 – Q3(c)

Let $X \neq 0$ be a complex Hilbert space and $T \in \mathscr{B}(X)$.
Suppose now $T$ is self-adjoint, $M = \sup V(T)$ and $m = \inf V(T)$.

1. Show that for any $w \in X$, $m\|w\|^2 \leq \langle Tw, w\rangle \leq M\|w\|^2$.
2. Take $\varepsilon > 0$ and $x$ be an element of $X$ with $\|x\| = 1$ such that $\langle Tx, x\rangle \geq M - \varepsilon$. Write $Tx$ as $\lambda x + y$ where $\lambda = \langle Tx, x\rangle$ and $\langle y, x\rangle = 0$. By considering elements of the form $w = x + ty$ where $t \in \mathbb{R}$, show that $\|y\|^2 \leq (M - m)\varepsilon$. Hence show that $M$ belongs to the approximate point spectrum of $T$.
3. State without a proof a formula for $\|T\|$ in terms of the inner product and hence express $\|T\|$ in terms of $m$ and $M$. Deduce that either $\|T\|$ or $-\|T\|$ belongs to $\sigma(T)$.

# Paper 2016 – Q3(c)(i)

Show that for any $w \in X$, $m\|w\|^2 \leq \langle Tw, w\rangle \leq M\|w\|^2$.

- As in (b)(ii), since $T$ is self-adjoints, $\langle Tw, w\rangle$ is real for all $w \in X$. It follows that $V(T)$ is non-empty and hence $M$ and $m$ are well-defined.
- If $w = 0$, the assertion is clear.
- Otherwise, let $x = w/\|w\|$ so that $\|x\| = 1$. Then $\langle Tx, x\rangle \in V(T)$ and so $m \leq \langle Tx, x\rangle \leq M$.
  Returning to $w$, we get $m\|w\|^2 \leq \langle Tw, w\rangle \leq M\|w\|^2$ as wanted.

# Paper 2016 – Q3(c)(ii)

Take $\varepsilon > 0$ and $x$ be an element of $X$ with $\|x\| = 1$ such that $\langle Tx, x\rangle \geq M - \varepsilon$. Write $Tx$ as $\lambda x + y$ where $\lambda = \langle Tx, x\rangle$ and $\langle y, x\rangle = 0$. By considering elements of the form $w = x + ty$ where $t \in \mathbb{R}$, show that $\|y\|^2 \leq (M - m)\varepsilon$. Hence show that $M$ belongs to the approximate point spectrum of $T$.

- Before going to the solution, let me remark that the assertion $M \in \sigma_{ap}(T)$ follows from what we did in the course: $T$ is self-adjoint so $\sigma(T) = \sigma_{ap}(T) \subset \mathbb{R}$, $\sigma(T) \subset [m, M]$ and both $m, M \in \sigma(T)$ , where to this last fact we use the fact that $rad(\sigma(T)) = \|T\|$ for a self-adjoint operator (which is itself a consequence of Gelfand's formula). This leg of the question gives a different way of proving this.

# Paper 2016 – Q3(c)(ii)

- Proceeding as instructed we take $x$ with $\|x\| = 1$ such that $\langle Tx, x\rangle \geq M - \varepsilon$.
- Let $\lambda = \langle Tx, x\rangle$ and $y = Tx - \lambda x$, we have that $\langle y, x\rangle = \langle Tx, x\rangle - \lambda\|x\|^2 = 0$.
- Now consider $w = x + ty$ for $t \in \mathbb{R}$. We have

$$\begin{aligned}\langle Tw, w\rangle &= \langle Tx, x\rangle + t^2\langle Ty, y\rangle + 2tRe\langle Ty, x\rangle \\ &\geq M - \varepsilon + t^2 m\|y\|^2 + 2tRe\langle y, \underbrace{Tx}_{=\lambda x + y}\rangle \\ &= M - \varepsilon + t^2 m\|y\|^2 + 2t\|y\|^2.\end{aligned}$$

- Using also $\langle Tw, w\rangle \leq M\|w\|^2 = M(1 + t^2\|y\|^2)$, we deduce that

$$(M - m)\|y\|^2 t^2 - 2t\|y\|^2 + \varepsilon \geq 0.$$

- Now if $M = m$, this is possible only if $y = 0$. If $M > m$, we choose $t = \frac{1}{M-m}$ and obtain $\|y\|^2 \leq (M - m)\varepsilon$ as wanted.

# Paper 2016 – Q3(c)(ii)

- Now, to show that $M \in \sigma_{ap}(T)$, we show that there exists $(x_n)$ with $\|x_n\| = 1$ such that $Tx_n - Mx_n \to 0$ as $n \to \infty$.
- Let $\varepsilon = 1/n$ in the previous computation, and relabel the corresponding $x$ as $x_n$, $y$ as $y_n$ and $\lambda$ as $\lambda_n$. We then have $M - \frac{1}{n} \leq \lambda_n \leq M$, $\|x_n\| = 1$.

$$Tx_n = \lambda_n x_n + y_n \text{ and } \|y_n\|^2 \leq \frac{M - m}{n}.$$

  Clearly this implies that $\|x_n\| = 1$ and $Tx_n - Mx_n = (\lambda_n - M)x_n + y_n \to 0$. We conclude that $M \in \sigma_{ap}(T)$.

# Paper 2016 – Q3(c)(iii)

State without a proof a formula for $\|T\|$ in terms of the inner product and hence express $\|T\|$ in terms of $m$ and $M$. Deduce that either $\|T\|$ or $-\|T\|$ belongs to $\sigma(T)$.

- Since $T$ is self-adjoint,

$$\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}.$$

- This implies that $\|T\| = \max\{|m|, |M|\} = \max\{-m, M\}$.
- Applying part (ii) to $T$, we have $M \in \sigma(T)$. Applying part (ii) to $-T$ instead, we have $-m \in \sigma(-T)$ and so $m \in \sigma(T)$.
- Now if $\|T\| = M$, we have $\|T\| = M \in \sigma(T)$. If $\|T\| = -m$, we have $-\|T\| = m \in \sigma(T)$.

# Paper 2008 – Q8(a)

Let $V$ be a complex Hilbert space, $T \in \mathscr{B}(V)$ is self-adjoint. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $v \in V$, prove that

$$\|\lambda v - Tv\| = \|\bar{\lambda} v - T^* v\| \geq |\mathrm{Im}\lambda| \|v\|. \tag{$\star$}$$

Deduce that $\lambda I - T$ and $(\lambda I - T)^*$ are injective. Prove that $\mathrm{Im}\,(\lambda I - T)$ is closed in $V$, and by considering the orthogonal complement $[(\lambda I - T)(V)]^\perp$, show that $\lambda I - T$ is surjective. Show that $\lambda I - T$ has a bounded inverse $(\lambda I - T)^{-1}$ with $\|(\lambda I - T)^{-1}\| \leq |\mathrm{Im}\lambda|^{-1}$. Deduce that $\sigma(T) \subset \mathbb{R}$.

# Paper 2008 – Q8(a)

- Suppose for the moment that $(\star)$ has been shown.
  It is clear that $(\star)$ implies that $\lambda I - T$ and $(\lambda I - T)^*$ are injective.
  Furthermore, this coercivity implies that the range $\mathrm{Im}\,(\lambda I - T)$ is closed (see Session 1).
  Since $[(\lambda I - T)(V)]^\perp = \mathrm{Ker}(\bar{\lambda} I - T^*) = 0$, we thus have that $\lambda I - T$ is surjective and hence bijective.
  Recalling $(\star)$ again, we have $\|(\lambda I - T)^{-1}\| \leq |\mathrm{Im}\lambda|^{-1}$, and so $\lambda \notin \sigma(T)$.
  Since $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is arbitrary, we have $\sigma(T) \subset \mathbb{R}$.
  So the main issue is to show $(\star)$.

- Switching the role of $\lambda$ and $\bar{\lambda}$, it is enough to estimate $\|\lambda v - Tv\|$.
- Write $\lambda = a + ib$. We compute

$$\begin{aligned}
\|\lambda v - Tv\|^2 &= \|av - Tv + ibv\|^2 \\
&= \|av - Tv\|^2 + b^2\|v\|^2 \\
&\qquad \underbrace{-ib\langle av - Tv, v\rangle + ib\langle v, av - Tv\rangle}_{=0 \text{ as } aI - T \text{ is self-adjoint}} \\
&= \|av - Tv\|^2 + b^2\|v\|^2 \geq |Im\lambda|^2\|v\|^2,
\end{aligned}$$

which proves $(\star)$.

# Paper 2010 – Q5

...Bookwork...

Let $X$ be a complex Hilbert space and $T \in \mathscr{B}(X)$ be self-adjoint and positive (i.e. $\langle Tx, x\rangle \geq 0$ for all $x \in X$). Prove that

(iii) $\sigma(T) \subset \mathbb{R}_{\geq 0}$.

(iv) $|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle$ for all $x, y \in X$.

Let $S, T \in \mathscr{B}(X)$ be self-adjoint and positive.

(v) Prove that if $x \in \mathrm{Ker}(S + T)$, then $\langle Tx, x\rangle = 0$.

(vi) Prove that if $Tx = Sx = 0$ for all $x \in \mathrm{Ker}(S + T)$.

(vii) Prove that if $ST = TS$ and $S^2 = T^2$, then $S = T$.

- The only spectral theory part is (iii), which is bookwork!
- One should probably use the adjective 'semi-positive' instead of 'positive'.

# Paper 2010 – Q5(iv)

Let $X$ be a complex Hilbert space and $T \in \mathscr{B}(X)$ be self-adjoint and positive (i.e. $\langle Tx, x\rangle \geq 0$ for all $x \in X$). Prove that

$$|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle \text{ for all } x, y \in X.$$

- This resembles Cauchy-Schwarz' inequality. The proof is the same. Here are some key steps:
- Replacing $y$ by $e^{i\theta}y$ for some suitable $\theta$, we may assume that $\langle Tx, y\rangle$ is real and non-negative.
- Now, for any $t \in \mathbb{R}$, we have

$$\begin{aligned}
0 &\leq \langle T(x+ty), x+ty\rangle \\
&= \langle Tx, x\rangle + t(\langle Tx, y\rangle + \langle Ty, x\rangle) + t^2\langle Ty, y\rangle \\
&= \langle Tx, x\rangle + t(\langle Tx, y\rangle + \langle y, Tx\rangle) + t^2\langle Ty, y\rangle && \text{(self-adjointness)} \\
&= \langle Tx, x\rangle + 2t\langle Tx, y\rangle + t^2\langle Ty, y\rangle && (\langle Tx, y\rangle \in \mathbb{R}).
\end{aligned}$$

# Paper 2010 – Q5(v)-(vi)

Let $X$ be a complex Hilbert space and $S, T \in \mathscr{B}(X)$ be self-adjoint and positive.
Prove that if $x \in \mathrm{Ker}(S + T)$, then $\langle Tx, x\rangle = 0$. Deduce that $Tx = 0$.

- Suppose $(S + T)x = 0$. Then $0 \leq \langle Tx, x\rangle = -\langle Sx, x\rangle \leq 0$. Hence $\langle Tx, x\rangle = 0$.
- Now take $y \in X$. By (iv) and the fact that $\langle Tx, x\rangle = 0$,

$$|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle = 0$$

  and so $\langle Tx, y\rangle = 0$. It follows that $Tx = 0$.

# Paper 2010 – Q5(vii)

Let $X$ be a complex Hilbert space and $S, T \in \mathscr{B}(X)$ be self-adjoint and positive.
Prove that if $ST = TS$ and $S^2 = T^2$, then $S = T$.

- Note that $0 = S^2 - T^2 = (S - T)(S + T)$. Thus $S - T$ is trivial in $\mathrm{Im}(S + T)$ and so in $\overline{\mathrm{Im}(S + T)} = \mathrm{Ker}(S + T)^\perp$.
- By (vi), $\mathrm{Ker}(S + T) \subseteq \mathrm{Ker} S \cap \mathrm{Ker} T \subseteq \mathrm{Ker}(S - T)$.
- Therefore $S - T$ vanishes on both $\mathrm{Ker}(S + T)^\perp$ and $\mathrm{Ker}(S + T)$. The conclusion follows from the projection theorem.