B4.2 Functional Analysis II
Lecture 1

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Tentative schedule of the course

- Hilbert spaces
- Baire category theorem and consequences
- Weak convergence and weak compactness
- Applications to the study Fourier series
- Spectral theory revisited

Things to note

- The lectures are typically less than 50 minutes in length.
- It is instructive to pause the video from time to time to contemplate your own proof before watching the proof of a statement.
- Occasionally, some statements in the video lectures and/or the lecture notes are left as exercise. It is important that you do them. They are statements which you are expected to be able to handle and are good for your development.
- Some lectures contain materials not written in the lecture notes.

Outline for the rest of the lecture

- Definition and examples of Hilbert spaces.
- Cauchy-Schwarz' inequality.
- Orthogonality in Hilbert spaces.
- Projection to closed convex sets in Hilbert spaces.

Real inner product spaces

Definition

An inner (scalar) product in a linear vector space X over $\mathbb{R}$ is a real-valued function on $X \times X$, denoted as $\langle x, y\rangle$, having the following properties:

1. *Bilinearity*. For fixed y, $\langle x, y\rangle$ is a linear function of x, and for fixed x, $\langle x, y\rangle$ is a linear function of y.
2. *Symmetry*. $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in X$.
3. *Posivity*. $\langle x, x\rangle > 0$ for $x \neq 0$.

$(X, \langle \cdot, \cdot \rangle)$ is called an inner product space.

Complex inner product spaces

Definition

An inner (scalar) product in a linear vector space X over $\mathbb{C}$ is a complex-valued function on $X \times X$, denoted as $\langle x, y\rangle$, having the following properties:

- (i') *Sesquilinearity*. For fixed y, $\langle x, y\rangle$ is a linear function of x, and for fixed x, $\langle x, y\rangle$ is a skewlinear function of y, i.e.

$$\langle ax, y\rangle = a\langle x, y\rangle \text{ and } \langle x, ay\rangle = \bar{a}\langle x, y\rangle \text{ for all } a \in \mathbb{C}, x, y \in X.$$

- (ii') *Skew symmetry*. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in X$.
- (iii') *Posivity*. $\langle x, x\rangle$ is real and positive for $x \neq 0$.

$(X, \langle\cdot,\cdot\rangle)$ is called an inner product space.

WARNING: In some textbooks, the sesquilinearity property is reversed: $\langle x, y\rangle$ is skewlinear in x and linear in y.

Inner product spaces as normed vector spaces

Suppose $(X, \langle \cdot, \cdot \rangle)$ is an inner product space. We define

$$\|x\| = \langle x, x \rangle^{1/2} \text{ for } x \in X.$$

Proposition

$\| \cdot \|$ *is a norm on* X*.*

Proof

- The positivity of the norm $\| \cdot \|$ is clear.
- The homogeneity of $\| \cdot \|$ follows from the bi/sesqui-linearity property.
- It remains to prove the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$. Squaring this and expanding, we see that this is equivalent to

$$\langle x, y \rangle + \langle y, x \rangle \leq 2\|x\|\|y\|.$$

  This is a consequence of Cauchy-Schwarz' inequality.

Cauchy-Schwarz' inequality

Theorem (Cauchy-Schwarz' inequality)

Suppose $(X, \langle \cdot, \cdot \rangle)$ *is an inner product space. For* $x, y \in X$,

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

Equality holds if and only if x *and* y *are linearly dependent.*

Proof

- The proof is the same as the proof in $\mathbb{R}^2$ or $\mathbb{C}^2$.
- If $y = 0$, the conclusion is clear. Assume henceforth that $y \neq 0$.
- Replacing x by ax with $|a| = 1$ so that $a\langle x, y \rangle$ is real, we may assume without loss of generality that $\langle x, y \rangle$ is real.
- Key: for $t \in \mathbb{R}$, we have

$$0 \leq \|x + ty\|^2 = \langle x + ty, x + ty \rangle = \|x\|^2 + 2t \, \mathrm{Re} \, \langle x, y \rangle + t^2 \|y\|^2.$$

Cauchy-Schwarz' inequality

Proof

- ...we may assume without loss of generality that $\langle x, y \rangle$ is real.
- $0 \leq \|x + ty\|^2 = \|x\|^2 + 2t \, \mathrm{Re} \, \langle x, y \rangle + t^2 \|y\|^2$ for all $t \in \mathbb{R}$.
- This implies that the discriminant of the quadratic polynomial on the right side is non-positive:

$$(\mathrm{Re} \, \langle x, y \rangle)^2 - \|x\|^2 \|y\|^2 \leq 0,$$

  This gives $|\mathrm{Re} \, \langle x, y \rangle| \leq \|x\| \|y\|$ which gives the desired inequality.
- If equality holds, then there is some t_0 such that $x + t_0 y = 0$, i.e. x and y are linearly dependent.

Definition

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. If the associated normed vector space $(X, \| \cdot \|)$ is complete, we call $(X, \langle \cdot, \cdot \rangle)$ a Hilbert space.

Example

- $\mathbb{R}^n$ is a Hilbert space with its usual dot product

$$\langle x, y \rangle := x \cdot y = \sum_{i=1}^{n} x_i y_i .$$

- $\mathbb{C}^n$ is a Hilbert space with its usual dot product

$$\langle x, y \rangle := x \cdot y = \sum_{i=1}^{n} x_i \bar{y}_i .$$

Hilbert spaces

Example

- $\ell^2(\mathbb{R})$ and $\ell^2(\mathbb{C})$ are Hilbert spaces with inner product

$$\langle x, y \rangle := \sum_{i=1}^{\infty} x_i y_i \text{ in the real case,}$$

$$\langle x, y \rangle := \sum_{i=1}^{\infty} x_i \bar{y}_i \text{ in the complex case.}$$

- Let (E, μ) be a measure space. Then $L^2(E, \mu)$ is a Hilbert space with inner product

$$\langle f, g \rangle := \int_E fg \, d\mu \text{ in the real case,}$$

$$\langle f, g \rangle := \int_E f\bar{g} \, d\mu \text{ in the complex case.}$$

Example

- If $(X, \langle \cdot, \cdot \rangle)$ is an inner product space and $Y \subset X$ is a vector subspace, then $(Y, \langle \cdot, \cdot \rangle)$ is an inner product space.
- If $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space and $Y \subset X$ is a vector subspace, then $(Y, \langle \cdot, \cdot \rangle)$ is a Hilbert space if and only if Y is closed.
- The space $C[0, 1]$ of real-valued continuous functions on $[0, 1]$ equipped with

$$\langle f, g \rangle = \int_0^1 f\, g\, dx$$

is an incomplete inner product space. Its completion is $L^2(0, 1)$.

Paralellogram law

Proposition (Paralellogram law)

If $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, then its associated norm satisfies

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \qquad (\dagger)$$

Conversely, if $(X, \| \cdot \|)$ is a normed vector space satisfying $(\dagger)$, then $\| \cdot \|$ is generated by an inner product on X.

Proof

- $(\Rightarrow)$ Direct computation.
- $(\Leftarrow)$ Sheet 1.

Paralellogram law

Example: $L^p(0,1)$, $1 \le p \le \infty$, is a Hilbert space if and only if $p = 2$.

- Let $f = \chi_{(0,1/2)}$ and $g = \chi_{(1/2,1)}$. Then $\|f\|_{L^p} = \|g\|_{L^p} = 2^{-1/p}$ and $\|f+g\|_{L^p} = \|f-g\|_{L^p} = 1$.
- The parallelogram law holds iff $2 = 4 \times 2^{-2/p}$ iff $p = 2$.

Orthogonality

Definition

Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space.

- Two vectors x and y in X are said to be orthogonal (perpendicular) if $\langle x, y\rangle = 0$.
- If Y is a subset of X, then $Y^\perp$ denotes the space of all vectors $v \in X$ which are orthogonal to Y, i.e. $\langle v, y\rangle = 0$ for all $y \in Y$. (Check that $Y^\perp$ is a vector space!)
  When Y is a subspace of X, $Y^\perp$ is called the orthogonal complement of Y in X.

Orthogonality

Proposition

Let Y be a subset of an inner product space $(X, \langle \cdot, \cdot \rangle)$. Then

1. (i) $Y^\perp$ *is a closed subspace of* X.
2. (ii) $Y \subset Y^{\perp\perp}$.
3. (iii) *If* $Y \subset Z \subset X$, *then* $Z^\perp \subset Y^\perp$.
4. (iv) $(\overline{Span\, Y})^\perp = Y^\perp$.
5. (v) *If* Y *and* Z *are subspaces of* X *such that* $X = Y + Z$ *and* $Z \subset Y^\perp$, *then* $Y^\perp = Z$.

We will only prove (i) and (v). The rest are left as exercise.

Orthogonality

Proof of (i):

- We are given that Y is a subset of X and we need to show that $Y^\perp$ is closed. (We already knew it is a subspace.)
- Let $(z_n) \subset Y^\perp$ and suppose $z_n \to z$. We then need to show $z \in Y^\perp$.
- Take an arbitrary $y \in Y$, we thus need to show $\langle y, z \rangle = 0$.
- As $z_n \in Y^\perp$, we know that $\langle y, z_n \rangle = 0$. So we would be done if we can show that $\langle y, z_n \rangle \to \langle y, z \rangle$.
- For this we use Cauchy-Schwarz' inequality:

$$|\langle y, z_n \rangle - \langle y, z \rangle| = |\langle y, z_n - z \rangle| \leq \|y\| \|z_n - z\| \to 0.$$

Piecing things together we are done.

Orthogonality

Proof of (v):

- We are given that Y, Z are subspaces of X such that $X = Y + Z$ and $Z \subset Y^\perp$. We need to show that $Z = Y^\perp$.
- Take $c \in Y^\perp$. We need to show $c \in Z$.
- Decompose $c = y + z$ where $y \in Y$ and $z \in Z$.
- Then $y = c - z$. The left side is an element of Y. The right side is an element of $Y^\perp$.
- So $\|c - z\|^2 = \langle y, c - z\rangle = 0$, which implies that $c = z$, i.e. $c \in Z$.

Y vs $Y^{\perp\perp}$

Remark

Let Y be a subspace of an inner product space X. We have that $Y \subset Y^{\perp\perp}$ and Y may be different from $Y^{\perp\perp}$.

Example

- If X is finite dimensional, then $Y^{\perp\perp} = Y$ (why?).
- Let $X = L^2(0,1)$ and $Y = C[0,1]$. Then $Y^{\perp} = 0$ (why?) and so $Y^{\perp\perp} = X \supsetneq Y$.

Example

Consider $X = L^2[0,1]$ as a real Hilbert space and let $Y = \mathbb{R}$ be the subspace of constant functions.

- The orthogonal complement of Y consists of function $f \in L^2(0,1)$ such that

$$0 = \langle f, 1 \rangle = \int_0^1 f\, dx.$$

  In other words, $Y^\perp$ consists of square integrable functions with zero average.
- If $f \in L^2(0,1)$, we can write f as the sum of it average $\bar{f}$ and $f - \bar{f}$. Clearly $\bar{f} \in Y$ and $f - \bar{f}$ has zero average and so $f - \bar{f} \in Y^\perp$. Hence $L^2(0,1) = Y^\perp + Y$.
- By part (v) of the proposition, $Y^{\perp\perp} = Y$.
- If $f \in Y \cap Y^\perp$, then f is constant and has zero average, so is zero. Hence $L^2(0,1) = Y^\perp \oplus Y$ as direct sum.

Projection to a closed convex set

Theorem (Closest point in a closed convex subset)

Let K be a non-empty closed convex subset of a Hilbert space X. Then, for every $x \in X$, there is a unique point $y \in K$ which is closer to x than any other points of K.

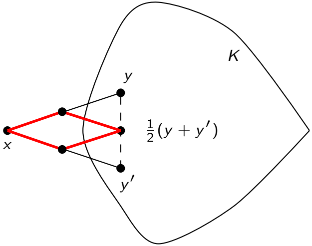
Proof

- For $z \in K$, let $d(z) = \|x - z\|$ and $d = \inf_K d$. The theorem asserts that d attains its minimum in K at a unique point.
- From Y1/Y2 analysis, we knew that if K is compact (i.e. K is closed and every sequence in K has a convergent subsequence), then d attains its minimum in K.
  But we don't know if K is compact (find easy examples to show this!), so we cannot argue this way.
  Instead, we draw crucially on the convexity of K.

Projection to a closed convex set

Proof

- For pedagogical purpose, let us consider first the uniqueness. Suppose $y, y' \in K$ are such that $d(y) = d(y') = \inf_K d$.
- By convexity $y'' = \frac{1}{2}(y + y') \in K$.



- Applying the parallelogram law to $\frac{1}{2}(y - x)$ and $\frac{1}{2}(y' - x)$ we get

$$d(y'')^2 + \frac{1}{4}\|y - y'\|^2 = 2\Big(\frac{1}{4}d(y)^2 + \frac{1}{4}d(y')^2\Big) = d^2.$$

- As $d \le d(y'')$, we thus have that $\|y - y'\|^2 = 0$, i.e. $y = y'$.

Projection to a closed convex set

Proof

- We now turn to existence. Let $y_n \in K$ be a minimizing sequence, i.e. $d_n := d(y_n) \to d$.
- We recycle the idea seen above: we apply the parallelogram law to $\frac{1}{2}(x - y_n)$ and $\frac{1}{2}(x - y_m)$ to get

$$d(\frac{1}{2}(y_n + y_m))^2 + \frac{1}{4}\|y_n - y_m\|^2 = \frac{1}{2}(d_n^2 + d_m^2).$$

- Since K is convex, $\frac{1}{2}(y_n + y_m) \in K$ and so $d \leq d(\frac{1}{2}(y_n + y_m))$. We thus have

$$d^2 + \frac{1}{4}\|y_n - y_m\|^2 = \frac{1}{2}(d_n^2 + d_m^2) \stackrel{m,n\to\infty}{\longrightarrow} d^2.$$

  Hence, $\|y_n - y_m\| \to 0$ as $m, n \to \infty$, i.e. (y_n) is Cauchy.
- Let $y = \lim y_n$. Since K is closed, $y \in K$. By the continuity of the norm, we have that $d(y) = \lim d(y_n) = d$ as wanted.