B4.2 Functional Analysis II - Sheet 2 of 4

Read the remaining of Chapter 1, Chapter 2 and prove the few statements whose proofs were left out as an exercise. (Not to be handed in.)

Do:

Q1. Let X be a Hilbert space and $A \in \mathscr{B}(X)$.

(a) Prove that $\text{Ker}\, A = (\text{Im}\, A^*)^\perp$ and $(\text{Ker}\, A)^\perp = \overline{\text{Im}\, A^*}$.

(b) Assume that A is a projection, i.e. $A^2 = A$. Show that $\text{Im}\, A$ is closed. Prove that

$$A = A^* \iff (\text{Im}\, A)^\perp = \text{Ker}\, A \iff \|A\| \le 1.$$

Deduce that either $\|A\| = 1$ or $A = 0$ provided that one of the above statements is true.

[*Hint: To prove that* $\|A\| \le 1$ *implies* $A = A^*$*, show that, for every given point in Im(*A*), the origin is the point in Im* $(I - A)$ *which is closest to that given point, and then use Q3 of Sheet 1 to show that Im* A *and Im(*$I - A$*) are orthogonal complementary spaces.*]

Q2. Let X be a Hilbert space and $U : X \to X$ be a unitary operator.

(a) Show that $\text{Ker}\,(I - U) = \text{Ker}\,(I - U^*)$;

(b) Show that $X = \overline{\text{Im}\,(I - U)} \oplus \text{Ker}\,(I - U)$;

(c) Show that $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=1}^{N-1} U^n x = x$ if $x \in \text{Ker}\,(I - U)$ and $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=1}^{N-1} U^n x = 0$ if $x \in \overline{\text{Im}\,(I - U)}$;

(d) Deduce that, for each $x \in X$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N-1} U^n x = Px,$$

where P is the orthogonal projection onto $\text{Ker}\,(I - U)$.

Q3. Let X be a Hilbert space and let $T \in \mathscr{B}(X)$.

(a) Prove that $\text{Ker}\, TT^* = \text{Ker}\, T^* = (\text{Im}\, T)^\perp$.

(b) Assume that T is normal, i.e. $T^*T = TT^*$. Prove that $\overline{\text{Im}\, T} = \overline{\text{Im}\, T^*}$.

(c) Prove that T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in X$.

Q4. Let M be a complete metric space and, for each $n \in \mathbb{N}$, let A_n be a nowhere dense subset of M and G_n be a dense open subset of M. Show that $\cap_{n\in\mathbb{N}} G_n$ is not contained in $\cup_{n\in\mathbb{N}} A_n$.

Deduce that $\mathbb{Q}$ is not the intersection of a countable number of open subsets of $\mathbb{R}$.

Q5. In this question, all sequence spaces are real.

(a) Consider a double sequence $(a_{n,j})$ such that for every fixed n, the sequence $(a_{n,j})_{j=1}^{\infty}$ belongs to c_0. Suppose that

$$\sup_n \sum_j a_{n,j}\, b_j < \infty \text{ for every } b = (b_j) \in \ell^1.$$

Show that $\sup_{n,j} |a_{n,j}| < \infty$.

(b) Suppose that (a_j) is a scalar sequence such that $\sum_j a_j b_j$ converges for all $b = (b_j) \in c_0$. Prove that $\sum_j |a_j|$ converges.

[*Hint: Consider the sequences T_n with entries $T_n(j) = a_j$ if $j \leq n$ and $T_n(j) = 0$ if $j > n$. Use the principle of uniform boundedness to show that (T_n) is bounded in ℓ^1.*]

(c*) (*Optional*) Let $2 < p < \infty$ and let $(c_{m,n})$ be a double sequence such that, for every fixed m,

$$\sum_n c_{m,n} a_n b_n \text{ converges for every } a = (a_n), b = (b_n) \in \ell^p$$

and

$$\sup_m \sum_n c_{m,n} a_n b_n < \infty \text{ for every } a = (a_n), b = (b_n) \in \ell^p.$$

Prove that, for $q = \frac{p}{p-2}$,

$$\sup_m \sum_n |c_{m,n}|^q < \infty.$$

Q6. (a) Let X be a real Banach space, Y and Z be real normed vector spaces, and $B : X \times Y \to Z$ be bilinear (i.e., linear in each variable). Suppose that for each $x \in X$ and $y \in Y$, the linear maps $B^x : Y \to Z$ and $B_y : X \to Z$ defined

$$B^x(y) = B(x, y) = B_y(x)$$

are continuous. Use the principle of uniform boundedness to prove that there exists a constant K such that $\|B(x, y)\| \leq K\|x\|\|y\|$ for all $x \in X$ and $y \in Y$. Deduce that B is continuous.

(b) Let X and Y both be the subspace of $L^1(0,1)$ consisting of polynomials, $Z = \mathbb{R}$, and

$$B(f,g) = \int_0^1 fg dt.$$

Show that the bilinear form B is continuous in each variables but it is not continuous.

[To put things in perspective, please note that even on $\mathbb{R}^2$, for nonlinear functions, separate continuity does not imply joint continuity. A standard example is the function $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq 0$ and $f(0,0) = 0$.]