Algebraic Number Theory: Problem Sheet 2. 2020/21.

Topics covered: integral bases, unique factorisation, Euclidean domains.

1. Show that $f(X) := X^3 - X + 2$ is an irreducible integer polynomial. Let θ be a root of $f(X) = 0$, and calculate the discriminant $\Delta^2(\Theta)$ of $\Theta := \{1, \theta, \theta^2\} \subset K := \mathbb{Q}(\theta)$. Using Stickelberger's theorem (Sheet 1 Question 7), show that Θ is an integral basis for K.

2. Let K be a number field and let $L = K(\sqrt{\delta})$, where $\delta \in K$ and $\sqrt{\delta} \notin K$. Let $\alpha = \beta + \gamma\sqrt{\delta} \in L$, where $\beta, \gamma \in K$. Show that if $\alpha \in \mathcal{O}_L$ then $\beta - \gamma\sqrt{\delta} \in \mathcal{O}_L$ and $\beta^2 - \gamma^2\delta \in \mathcal{O}_K$.

3. Let R be an integral domain. Show that every prime element in R is irreducible. Suppose now that factorisation into irreducible elements in R is possible. Show that this factorisation is unique if and only if every irreducible element in R is prime.

4. Let K be a number field. Prove that $\mathcal{O}_K$ is Euclidean with norm function $|\mathrm{Norm}_{K/\mathbb{Q}}| : \mathcal{O}_K \backslash \{0\} \to \mathbb{N} \cup \{0\}$ if and only if for each $\alpha \in K$ there exists $\beta \in \mathcal{O}_K$ such that $|\mathrm{Norm}_{K/\mathbb{Q}}(\alpha - \beta)| < 1$. Verify this condition for $K = \mathbb{Q}(\sqrt{-d})$ where $d = 1, 2, 3, 7$. [Hint: Consider the nearest point on a lattice.]

5. Prove that if p is a prime with $p \equiv 1$ or $3 \bmod 8$, then $p = X^2 + 2Y^2$ for some $X, Y \in \mathbb{Z}$ unique up to sign.

6. Show that $\mathbb{Z}[\sqrt{3}]$ is Euclidean. Write down the factorisations into irreducibles of 2, 3 and 11. Show using congruences modulo 3 and 4 that the elements 5 and 7 are irreducible in $\mathbb{Z}[\sqrt{3}]$.

7. Show that $X^2 - 10Y^2 = \pm 2$ is insoluble in integers. Deduce that if α divides both 2 and $\sqrt{10}$ in $\mathbb{Z}[\sqrt{10}]$, then α is a unit. Show however that one cannot write $\alpha = 2\beta + \sqrt{10}\gamma$ with $\beta, \gamma \in \mathbb{Z}[\sqrt{10}]$.

8. Let K be a number field and let A, B, C, I be ideals of $\mathcal{O}_K$ Recall that A, B are said to be *coprime* if $A + B = \mathcal{O}_K$. Show that if A, B are coprime and $A|BC$ then $A|C$. Show that if A, B are coprime and $A|I, B|I$ then $AB|I$.

Further Practice: Exercises in Chapter 3,4 and 5 of Stewart and Tall.