Algebraic Number Theory: Problem Sheet 4. 2020/21.

Topics covered: Minkowski's convex body theorem; calculation of class numbers; Diophantine applications.

1. Let $d \neq 0, 1$ be a square-free integer and p a prime. Let $K = \mathbb{Q}(\sqrt{d})$ and denote by $\Delta^2 := \Delta^2(K)$ the discriminant of K. Show that

   - (p) ramifies in K if and only if p divides Δ^2,
   - (p) splits as a product of two distinct prime ideals in $\mathcal{O}_K$ if and only if p is odd and $\left(\frac{d}{p}\right) = 1$, or $p = 2$ and $d \equiv 1 \bmod 8$.
   - (p) is still prime in $\mathcal{O}_K$, (i.e. (p) is inert), if and only if p is odd and $\left(\frac{d}{p}\right) = -1$, or $p = 2$ and $d \equiv 5 \bmod 8$.

   Here $\left(\frac{d}{p}\right)$ is the Legendre symbol.

2. Let $K = \mathbb{Q}(\sqrt[3]{6})$. Given that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{6}]$, find the prime factorisation of the ideals $(2), (3), (5)$ and (7) in $\mathcal{O}_K$. Show that all prime ideal factors which occur are principal. Using Minkowski's bound, deduce that $\mathcal{O}_K$ is a PID.

3. Let $\sigma : K \to K$ be an automorphism. Let $I = (\alpha_1, \ldots, \alpha_n)$, an ideal of $\mathcal{O}_K$. Show that $I^\sigma = (\alpha_1^\sigma, \ldots, \alpha_n^\sigma)$. Show that, if I is nonzero then $N(I^\sigma) = N(I)$. Show that $N\big((3, 1+\sqrt{7})\big) = N\big((3, 1-\sqrt{7})\big)$ in $\mathcal{O}_{\mathbb{Q}(\sqrt{7})}$. Show that $N\big((3, 1+\sqrt{2}+\sqrt{3}+\sqrt{6})\big) = N\big((3, 1-\sqrt{2}-\sqrt{3}+\sqrt{6})\big)$ in $\mathcal{O}_{\mathbb{Q}(\sqrt{2},\sqrt{3})}$.

4. Suppose that a prime p does not divide the class number of a number field K. Show that if I is a non-zero ideal of $\mathcal{O}_K$, and I^p is principal, then I is principal.

5. Show that there are no integer solutions to the equation $Y^2 = X^3 - 5$.

6. Calculate the class number h_K for $K := \mathbb{Q}(\sqrt{-23})$.

Further Practice: Exercises in Chapters 7,8,9 and 10 of Stewart and Tall.