B3.3 Algebraic Curves

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(Questions on lectures 1-3)

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Example sheet 1

1. Embed $\mathbb{R}^2$ in the projective plane $\mathbb{RP}^2$ by $(x, y) \mapsto [1, x, y]$. Find the point of intersection in $\mathbb{RP}^2$ of the projective lines corresponding to the parallel lines $y = mx$ and $y = mx + c$ in $\mathbb{R}^2$.

2. Let $\mathbb{Z}_2$ be the field $\{0, 1\}$. Show that the number of points in n-dimensional projective space over $\mathbb{Z}_2$ is $2^{n+1} - 1$. How many projective lines are there in this space?

3. What are the answers in Q2 if you instead work over the field $\mathbb{Z}_p$ with p elements, where p is prime?

4. Show that

$$f : ([z_0, z_1], [w_0, w_1]) \mapsto [z_0 w_0, -z_0 w_1 - z_1 w_0, z_1 w_1]$$

is a well-defined map from $\mathbb{CP}^1 \times \mathbb{CP}^1$ to $\mathbb{CP}^2$.

Also show that this map is surjective.

Is the corresponding map $f : \mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{RP}^2$ surjective?

5. Adapt the ideas from the lectures to show that complex projective space $\mathbb{CP}^n$ is compact. What is the relationship between this space and a sphere of appropriate dimension?

6. Prove Pappus's theorem by using the general position ideas outlined in lectures. First prove the theorem in the degenerate case when A, B, C', B' are *not* in general position. Then assume these points *are* in general position and take them to be $[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1]$. Calculate the three intersections explicitly, verify they are collinear, and explain why this proves the theorem in general.