B3.3 Algebraic Curves

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(questions on lectures 4-6)

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Example sheet 2

1. Let C be the projective curve with equation

$$x^2 + y^2 = z^2.$$

Show that the projective line through the points $[0, 1, 1]$ and $[t, 0, 1]$ meets C in the two points $[0, 1, 1]$ and $[2t, t^2 - 1, t^2 + 1]$.

Show that there is a bijection between the projective line $y = 0$ and C given by:

$$[t, 0, 1] \mapsto [2t, t^2 - 1, t^2 + 1]$$

$$[1, 0, 0] \mapsto [0, 1, 1]$$

2. Show that a homogeneous polynomial in two variables x, y may be factored into linear polynomials over $\mathbb{C}$.

3. This question deals with how to define tangent lines at singular points.

Let C be a curve in $\mathbb{C}^2$ defined by $Q(x, y) = 0 \ : x, y \in \mathbb{C}$. Define the *multiplicity* m of C at a point $(a, b) \in C$ to be the smallest positive integer m such that some mth partial derivative of Q at (a, b) is nonzero (so (a, b) is a singularity of C iff $m > 1$).

Consider the polynomial

$$\sum_{i+j=m} \frac{\partial^m Q}{\partial x^i \partial y^j}(a, b) \frac{(x - a)^i (y - b)^j}{i! j!}.$$

As in question 2, we can factorise this as a product of m linear polynomials of the form

$$\alpha(x - a) + \beta(y - b).$$

The lines defined by the vanishing of these linear polynomials are called the m tangent lines to C at (a, b).

(i) Show that if $m = 1$ this definition agrees with the definition given in lectures for the tangent line at a nonsingular point.

(ii) Find the multiplicities and tangent lines of the singularities for the nodal cubic $y^2 = x^3 + x^2$ and the cuspidal cubic $y^2 = x^3$.

4. Show that if $\alpha_1, \ldots, \alpha_r$ are *distinct*, then the affine curve

$$y^2 = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_r)$$

is nonsingular.

What can you say about the associated projective curve?

5. (i) Show that the affine curve $y^2 = x^3 + x$ in $\mathbb{C}^2$ is nonsingular.

(ii) Now consider this curve over the finite field $\mathbb{Z}_p$ where p is a prime. That is, we consider the curve in $(\mathbb{Z}_p)^2$ with equation $y^2 = x^3 + x$.

For which p is this nonsingular?