B3.3 Algebraic Curves

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(questions on lectures 6-11)

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Example sheet 3

1. (i) Show that given any 5 points in $\mathbb{CP}^2$, there is at least one conic passing through them. Show also that this conic is unique if no three of the points are collinear.

(ii) Let C be a quartic (degree 4) curve in $\mathbb{CP}^2$ with four singular points. Use the strong form of Bézout's theorem to show C must be reducible.

(iii) Show that $y^4 - 4xzy^2 - xz(x-z)^2 = 0$ defines a quartic with three singular points.

2. Let $P(x, y, z)$ be a homogeneous polynomial of degree d defining a nonsingular curve C.

(i) Write down Euler's relation for P, P_x, P_y, P_z. Deduce that the Hessian determinant satisfies:

$$z\mathcal{H}_P(x,y,z) = (d-1)\det\begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_x & P_y & P_z \end{pmatrix}.$$

(ii) Deduce further that:

$$z^2\mathcal{H}_P(x,y,z) = (d-1)^2\det\begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & P_y \\ P_x & P_y & dP/(d-1) \end{pmatrix}.$$

(iii) Deduce that if $P(x, y, 1) = y - g(x)$ then $[a, b, 1]$ is a flex of C iff $b = g(a)$ and $g''(a) = 0$.

3. Let C and D be nonsingular projective curves of degree n and m in $\mathbb{P}^2$. Show that if C is homeomorphic to D then either $n = m$ or $\{n, m\} = \{1, 2\}$.

4. Show that if C is the conic $y^2 = xz$ then the map

$$f : \mathbb{P}^1 \to C$$

given by

$$f : [s, t] = [s^2, st, t^2]$$

is a homeomorphism.

Deduce without using the degree-genus formula that all nonsingular conics have genus zero.

5. Let $f : X \to Y$ be a (nonconstant) holomorphic map of compact connected Riemann surfaces, where X is the Riemann sphere. Show that Y is homeomorphic to X.

6(i) Let U be a connected open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be holomorphic. Show that if $a \in U$, then for sufficiently small real positive r, we have:

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

(ii) Deduce that if $|\ f\ |$ has a local maximum at $a \in U$, then $|\ f\ |$ is constant on some neighbourhood of a.

(iii) Deduce that if $|\ f\ |$ has a local maximum at $a \in U$, then f is constant on U.

(iv) Now suppose S is a compact connected Riemann surface and $f : S \mapsto \mathbb{C}$ is a holomorphic function. Show that f is constant.(You may assume the Identity Theorem for Riemann surfaces, that is, if two holomorphic maps on a Riemann surface agree then on a nonempty open set then they agree everywhere).