Algebraic Curves

Section B course Hilary 2021

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Example sheet 4

1. Let $\Lambda = \{m\omega_1 + n\omega_2 \ : \ m, n \in \mathbb{Z}\}$ be a lattice in $\mathbb{C}$ and let f be meromorphic and doubly periodic with respect to Λ.

Let $\Gamma(a)$ denote the solid parallelogram with vertices at $a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2$ and let $\gamma(a)$ be the boundary of $\Gamma(a)$. Choose a so that f has no zeroes or poles on $\gamma(a)$.

Let $\beta_1, \ldots, \beta_s$ denote the set of poles of f inside $\gamma(a)$.

Show that

$$\sum_{i=1}^{s} \mathrm{Res}(f; \beta_i) = 0.$$

2. Consider the affine nodal cubic C_{aff} in $\mathbb{C}^2$ with equation

$$y^2 = x^3 + x^2.$$

Show that the formula

$$t \mapsto (t^2 - 1, t - t^3)$$

describes a map from $\mathbb{C}$ onto C_{aff}. Describe the fibres of this map (ie. the preimages of points in C_{aff}).

What can you deduce about the topology of the projective nodal cubic $y^2 z = x^3 + x^2 z$?

3. Let $\wp(z)$ be the Weierstrass $\wp$-function associated to a lattice Λ. Consider the meromorphic function $\wp'(z)$ as a function from the elliptic curve $X = \mathbb{C}/\Lambda$ to the Riemann sphere.

Determine its degree and the number and ramification indices of its ramification points.

Is there a meromorphic function f on X with $f'(z) = \wp(z)$?

4. Let E be an elliptic curve, that is, a Riemann surface of genus 1. and let p be a point on E.

Calculate $\ell(mp)$ for $m = 1, 2, 3, \ldots$.

Deduce that there exist meromorphic functions f and g on E with, respectively, a double pole at a and a triple pole at a, and no other poles.

Describe $\mathcal{L}(mp)$ for $m = 1, 2, 3, 4, 5$ in terms of the functions f and g.

By considering $\mathcal{L}(6p)$, deduce that we have a polynomial relation between f and g, and interpret your results in terms of the Weierstrass $\wp$-function.