Sheet 2. Prerequisites: sections 1-8. Week 5

Q1. Consider the ideals $\mathfrak{p}_1 := (x, y)$, $\mathfrak{p}_2 := (x, z)$ and $\mathfrak{m} := (x, y, z)$ of $K[x, y, z]$, where K is a field. Show that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{p}_1 \cdot \mathfrak{p}_2$. Determine the isolated and the embedded prime ideals of $\mathfrak{p}_1 \cdot \mathfrak{p}_2$.

Q2. Let K be a field. Show that the ideal $(x^2, xy, y^2) \subseteq K[x, y]$ is a primary ideal, which is not irreducible.

Q3. Let R be a noetherian ring and let T be a finitely generated R-algebra. Let G be a finite subgroup of the group of automorphisms of T as a R-algebra. Let T^G be the fixed point set of G (ie the subset of T, which is fixed by all the elements of G). Show that T^G is a subring of T, which contains R and that T^G is finitely generated over R.

Q4. Show that $\mathbb{Z}$ is integrally closed and that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$ is $\mathbb{Z}[i]$.

Q5. Let S be a ring and let $R \subseteq S$ be a subring of S. Suppose that R is integrally closed in S. Let $P(x) \in R[x]$ and suppose that $P(x) = Q(x)J(x)$, where $Q(x), J(x) \in S[x]$ and $Q(x)$ and $J(x)$ are monic. Show that $Q(x), J(x) \in R[x]$. Use this to give a new proof of the fact that if $T(x) \in \mathbb{Z}[x]$ and $T(x) = T_1(x)T_2(x)$, where $T_1(x), T_2(x) \in \mathbb{Q}[x]$ are monic polynomials, then $T_1(x), T_2(x) \in \mathbb{Z}[x]$.

Q6. Let R be a subring of a ring T and suppose that T is integral over R. Let $\mathfrak{p}$ be prime ideal of R and let $\mathfrak{q}$ be a prime ideal of T. Suppose that $\mathfrak{q} \cap R = \mathfrak{p}$. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_k$ be primes ideal of R and suppose that $\mathfrak{p}_1 = \mathfrak{p}$. Show that there are prime ideals $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_k$ of T such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for all $i \in \{1, \ldots, k\}$.

Q7. Let R be a ring. Let $\mathcal{S}$ be the set of ideals in R, which are not finitely generated.

(i) Let I be maximal element of $\mathcal{S}$ (with respect to the relation of inclusion). Show that I is prime.

(ii) Suppose that all the prime ideals of R are finitely generated. Prove that R is noetherian.

[Hint: exploit the fact that R/I is noetherian.]

Q8. (optional). Let R be a ring. Let $\mathcal{S}$ be the set of non-principal ideals in R. Let I be a maximal element of $\mathcal{S}$. Prove that I is a prime ideal.