Sheet 1. Prerequisites: sections 1-5. Week 3.

Q1. Let R be a ring. Show that the Jacobson radical of R coincides with the set $\{x \in R \,|\, 1 - xy \text{ is a unit for all } y \in R\}$.

Q2. Let R be a ring.

(i) Show that if $P(x) = a_0 + a_1 x + \cdots + a_k x^k \in R[x]$ is a unit of $R[x]$ then a_0 is a unit of R and a_i is nilpotent for all $i \geq 1$.

(ii) Show that the Jacobson radical and the nilradical of $R[x]$ coincide.

Q3. Let R be a ring and let $N \subseteq R$ be its nilradical. Show that the following are equivalent:

(i) R has exactly one prime ideal.

(ii) Every element of R is either a unit or is nilpotent.

(iii) R/N is a field.

Q4. Let R be a ring and let $I \subseteq R$ be an ideal. Let $S := \{1 + r \,|\, r \in I\}$.

(i) Show that S is a multiplicative set.

(ii) Show that the ideal generated by the image of I in R_S is contained in the Jacobson radical of R_S.

(iii) Prove the following generalisation of Nakayama's lemma:

Lemma. *Let M be a finitely generated R-module and suppose that $IM = M$. Then there exists $r \in R$, such that $r - 1 \in I$ and such $rM = 0$.*

Q5. Let R be a ring and let M be a finitely generated R-module. Let $\phi : M \to M$ be a surjective homomorphism of R-modules. Prove that ϕ is injective, and is thus an automorphism. [Hint: use ϕ to construct a structure of $R[x]$-module on M and use the previous question.]

Q6. Let R be a ring. Let $\mathcal{S}$ be the subset of the set of ideals of R defined as follows: an ideal I is in $\mathcal{S}$ iff all the elements of I are zero-divisors. Show that $\mathcal{S}$ has maximal elements (for the relation of inclusion) and that every maximal element is a prime ideal. Show that the set of zero divisors of R is a union of prime ideals.

Q7. Let R be a ring. Consider the inclusion relation on the set $\mathrm{Spec}(R)$. Show that there are minimal elements in $\mathrm{Spec}(R)$.