Sheet 3. Prerequisites: sections 1-10. Week 7

Q1. Let R be a subring of a ring T. Suppose that T is integral over R. Let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{q}_1, \mathfrak{q}_2$ be prime ideals of T such that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R = \mathfrak{p}$ and $\mathfrak{q}_1 \neq \mathfrak{q}_2$. Show that we have $\mathfrak{q}_1 \not\subseteq \mathfrak{q}_2$ and $\mathfrak{q}_2 \not\subseteq \mathfrak{q}_1$.

Q2. Let R be a ring. Show that the two following conditions are equivalent:

(i) R is a Jacobson ring.

(ii) If $\mathfrak{p} \in \mathrm{Spec}(R)$ and $R/\mathfrak{p}$ contains an element b such that $(R/\mathfrak{p})[b^{-1}]$ is a field, then $R/\mathfrak{p}$ is a field.

Here we write $(R/\mathfrak{p})[b^{-1}]$ for the localisation of $R/\mathfrak{p}$ at the multiplicative subset $1, b, b^2, \ldots$.

Q3. Let k be field and let R be a finitely generated algebra over k. Show that the two following conditions are equivalent:

(i) $\mathrm{Spec}(R)$ is finite.

(ii) R is finite over k.

Q4. Let k be an algebraically closed field. Let $P_1, \ldots, P_d \in k[x_1, \ldots, x_d]$. Suppose that the set

$$\{(y_1, \ldots, y_d) \in k^d \,|\, P_i(y_1, \ldots, y_d) = 0 \,\forall i \in \{1, \ldots, d\}\}$$

is finite. Show that

$$\mathrm{Spec}(k[x_1, \ldots, x_d]/(P_1, \ldots, P_d))$$

is finite.

Q5. Let R be a ring and let R_0 be the prime ring of R (see the preamble of the notes for the definition). Suppose that R is a finitely generated R_0-algebra. Suppose also that R is a field. Prove that R is a finite field.

Q6. Let k be a field and let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_d]$. Show that there are polynomials $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \ldots, P_d(x_1, \ldots, x_d)$ such that $\mathfrak{m} = (P_1, \ldots, P_d)$.

Q7. Let R be a domain. Show $R[x]$ is integrally closed if R is integrally closed.

Here are some hints for this exercise. Let K be the fraction field of R.

(i) Show first that it suffices to show that $R[x]$ is integrally closed in $K[x]$ (ie that the integral closure of $R[x]$ in $K[x]$ is $R[x]$).

(ii) Consider $Q(x) \in K[x]$ and suppose that $Q(x)$ is integral over $R[x]$. Show that $Q(x) + x^t$ satisfies an integral equation with coefficients in $R[x]$, whose constant coefficient is a monic polynomial, if t is sufficiently large.

(iii) Conclude.