Exercise sheet 4. Prerequisites: all lectures. W1 of Trinity Term

Q1. Let R be a noetherian domain. Let $\mathfrak{m}$ be a maximal ideal in R. Let $r \in R\backslash\{0\}$ and suppose that (r) is a $\mathfrak{m}$-primary ideal. Show that $\mathrm{ht}((r)) = 1$.

Q2. Let A, B be integral domains and suppose that $A \subseteq B$. Suppose that A is integrally closed and that B is integral over A. Let

$$\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n$$

be a descending chain of prime ideals in A. Let $k \in \{0, \dots, n-1\}$ and let

$$\mathfrak{q}_0 \supsetneq \mathfrak{q}_1 \supsetneq \cdots \supsetneq \mathfrak{q}_k$$

be a descending chain of prime ideals in B, such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \in \{0, \dots, k\}$. Then there is a descending chain of prime ideals

$$\mathfrak{q}_k \supsetneq \mathfrak{q}_{k+1} \supsetneq \cdots \supsetneq \mathfrak{q}_n$$

such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \in \{k+1, \dots, n\}$. This is the "going-down theorem". See AT, Th. 5.16, p. 64. Let L (resp. K) be the fraction field of B (resp. A). Prove the going-down theorem when L is a (finite) Galois extension of K.

Q3. Let R be an integrally closed domain. Let $K := \mathrm{Frac}(R)$. Let $L|K$ be an algebraic field extension. Show that an element $e \in L$ is integral over R iff the minimal polynomial of e over K has coefficients in R.

Q4. Let R be a PID. Suppose that $2 = 1 + 1$ is a unit in R. Let $c_1, \dots, c_t \in R$ be distinct irreducible elements and let $c := c_1 \cdots c_t$. Show that the ring $R[x]/(x^2 - c)$ is a Dedekind domain. Use this to show that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is a Dedekind domain.

Q5. Let R be a PID. Suppose that $2 = 1 + 1$ is invertible in R. Let $c_1, c_2 \in R$ be two distinct irreducible elements and let $c := c_1 \cdot c_2$. Show that the decomposition of the ideal (c) in $R[x]/(x^2 - c)$ into a product of prime ideals is $(c) = (x, c_1)^2 \cdot (x, c_2)^2$ (noting that $R[x]/(x^2 - c)$ is a Dedekind domain by Q4).

Q6. Let R be a ring (not necessarily noetherian). Suppose that $\dim(R) < \infty$.

Show that $\dim(R[x]) \leq 1 + 2\dim(R)$.

Q7. Let R be a Dedekind domain. Let $\mathfrak{a}$ be a non zero ideal in R. Show that every ideal in $R/\mathfrak{a}$ is principal. Show that every ideal in a Dedekind domain can be generated by two elements.

Q8. (optional) Let A (resp. B) be a local ring with maximal ideal $\mathfrak{m}_A$ (resp. $\mathfrak{m}_B$). Let $\phi : A \to B$ be a ring homomorphism and suppose that $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ (such a homomorphism is said to be 'local').

Suppose that

(1) B is finite over A via ϕ;

(2) the map $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ induced by ϕ is surjective;

(3) the map $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ induced by ϕ is bijective.

Prove that ϕ is surjective. [Hint: use Nakayama's lemma twice].

Q9. (optional) Let R be a Dedekind domain. Show that R is a PID iff it is a UFD.