

## Problem Set 4

1. (a) Let  $X$  be a well-ordered set, and  $x \in X$ . Show that either  $x$  is the greatest element in  $X$  or  $x$  has an immediate successor (that is an element  $x^* \in X$  with  $x < x^*$  such that there is no  $y \in X$  with  $x < y < x^*$ ).

(b) Let  $X \subset \mathbb{R}$  such that the inherited order  $<$  from  $\mathbb{R}$  is a well-order on  $X$ . Prove that  $X$  must be countable. [Hint: consider the intervals  $(x, x^*)$ .]

2. Let  $<_A, <_B$  be strict total orders on sets  $A, B$  respectively. We define the *sum*  $(A, <_A) + (B, <_B)$  and the *product*  $(A, <_A) \times (B, <_B)$  of the orders as follows.

For the sum, we assume  $A, B$  are disjoint (which can always be arranged by replacing them by  $A' = \{0\} \times A, B' = \{1\} \times B$  with the obvious orders on them). Then  $(A, <_A) + (B, <_B)$  is the set  $A \cup B$  with the order  $<_+$  in which elements of  $A$  or  $B$  are ordered by  $<_A, <_B$  respectively and all elements of  $A$  precede all elements of  $B$ .

The product  $(A, <_A) \times (B, <_B)$  is  $A \times B$  with the *reverse lexicographic order*, that is  $(a, b) <_\times (a', b')$  iff  $b < b'$ , or  $b = b'$  and  $a < a'$ .

(i) Draw illustrative pictures (coloured pens may be helpful) of the orders

$$\omega + 4, \quad 4 + \omega, \quad \omega + \omega, \quad \omega \cdot \omega$$

(ii) Prove that  $<_+, <_\times$  are well-orders if  $<_A, <_B$  are well-orders. [You may omit the (tedious) verification that they are strict orders and that they are total.]

3. Let  $\alpha, \beta, \gamma$  be ordinals. Show that

- (i) if  $\beta < \gamma$  then  $\alpha + \beta < \alpha + \gamma$  (Hint: induction on  $\gamma$ , or use Theorem 14.7).
- (ii) if  $\alpha + \beta = \alpha + \gamma$  then  $\beta = \gamma$ , i.e. left cancellation holds.
- (iii) right cancellation “ $\alpha + \gamma = \beta + \gamma$  implies  $\alpha = \beta$ ” fails, by giving a counterexample.
- (iv) if  $\gamma$  is a limit ordinal then  $\alpha + \gamma$  is a limit ordinal.

4. For any two ordinals  $\alpha, \beta$ , exactly one of  $\alpha \in \beta, \alpha = \beta, \beta \in \alpha$  hold. Determine which of these holds when

- (i)  $\alpha = (\omega + 1).2, \quad \beta = 2.(\omega + 1)$
- (ii)  $\alpha = (\omega + 1).\omega, \quad \beta = \omega.(\omega + 1)$

5. Ordinal exponentiation  $\beta \mapsto \alpha^\beta$  for any ordinal  $\alpha > 0$  was defined in lectures. Prove that if  $\alpha, \beta$  are countable, with  $\alpha > 0$ , then  $\alpha^\beta$  is countable. [Observe the difference with cardinal exponentiation on this point]. Assume AC. It can be done without with more work.

6. Let  $P$  be a non-empty partially strictly ordered set and assume no element of  $P$  is maximal (i.e. for every  $x \in P$  there exists  $y \in P$  with  $x < y$ ). Use AC to show there exists a function  $f : \omega \rightarrow P$  with  $f(n) < f(n^+)$  for all  $n \in \omega$ .

7. Let  $R$  be a commutative ring with identity  $1 \neq 0$ .

- (i) Prove that the union of a chain of proper ideals is a proper ideal.
- (ii) Use Zorn's Lemma to prove that  $R$  has a maximal ideal.

8. Let  $\mathcal{A}$  be a non-empty set of non-empty sets and  $X = \bigcup \mathcal{A}$ . Assume Zorn's Lemma and prove the existence of a function  $F : \mathcal{A} \rightarrow X$  with  $F(A) \in A$  for each  $A \in \mathcal{A}$ . [Hint: apply ZL to the partially ordered set  $P$  of partial maps from  $\mathcal{A}$  to  $X$ , regarded as a subset of  $\mathcal{P}(\mathcal{A} \times X)$ , ordered by inclusion.]

Deduce that Zorn's Lemma implies the Axiom of Choice.