B1 Set Theory: Problem sheet 2

1. (i) Let R be a relation (that is, set of ordered pairs). Prove that $\operatorname{dom} R$, which we define to be $\{x : \exists y\,(\langle x, y\rangle \in R)\}$, is a set.

(ii) Let a and b be sets. Prove that there exists a set whose members are exactly the functions with domain a and codomain b.

2. Let a be a set. Prove that $\{a\} \times \{a\} = \{\{\{a\}\}\}$.

3. Show that if we define the ordered triple $\langle x, y, z\rangle$ to be $\big\langle \langle x, y\rangle, z\big\rangle$, then this definition works in the sense that $\langle u, v, w\rangle = \langle x, y, z\rangle$ if and only if $u = x$, $v = y$, and $w = z$.

For each of the following possible definitions of an ordered triple, prove that it works, or give a counterexample.

(i) $\langle x, y, z\rangle_1 = \big\{\{x\}, \{x, y\}, \{x, y, z\}\big\}$;

(ii) $\langle x, y, z\rangle_2 = \big\{\langle 0, x\rangle, \langle 1, y\rangle, \langle 2, z\rangle\big\}$;

(iii) $\langle x, y, z\rangle_3 = \big\langle \{0, x\}, \{1, y\}, \{2, z\}\big\rangle$.

4. Use the Axiom of Foundation to show that if A is a non-empty set, then $A \neq A \times A$. [*Hint: consider the set* $a = A \cup \bigcup A$.]

5. Prove that $\forall x\, \exists y\, y \notin x$.

6. Prove that there is no function f with domain ω such that $f(n^+) \in f(n)$ for all $n \in \omega$. [Hint: apply the Axiom of Foundation to $\operatorname{ran} f$.] Deduce that, for any set x, it is false that $x \in x$.

7. Prove, using the Principle of Induction and the fact that each $n \in \omega$ is a transitive set, that $n \in n$ is false for every natural number n. [*That is, do not use the Axiom of Foundation.*]

8. Give proofs by induction of the following, for m, $n \in \omega$:

(i) $0.n = 0$,

(ii) $m^+.n = m.n + n$,

(iii) $m.n = n.m$.

[*You may assume any properties of addition that you need. Note that* (a) *and* (b) *would be immediate from the definition of multiplication if we assumed that multiplication is commutative. But* (a) *and* (b) *are stepping stones to* (c).]

9. Write $1 = 0^+$ and $2 = 1^+$, and define $n \in \omega$ to be even if it is of the form $2.p$ for some $p \in \omega$ and odd if it is of the form $2.q + 1$ for some $q \in \omega$.

(i) Prove that every natural number is either even or odd.

(ii) Prove that no $n \in \omega$ is both even and odd.

10. Suppose that $\langle A, s, a_0\rangle$ is a *Peano system*, that is, A is a set, $a_0 \in A$, and $s : A \to A$ is a function which (a) is one-to-one, (b) does not include a_0 in its range, and (c) satisfies that Principle of Induction: that is, if $S \subseteq A$, $a_0 \in S$, and $\forall a\,(a \in S \to s(a) \in S)$, then $S = A$.

Prove that there exists an isomorphism from $\langle \omega, {}^+, 0\rangle$ to $\langle A, s, a_0\rangle$; that is, there is a bijection $f : \omega \to A$ such that $f(0) = a_0$, and for all $n \in \omega$, $f(n^+) = s(f(n))$. [*Hence, up to isomorphism,* $\langle \omega, {}^+, 0\rangle$ *is the only Peano system.*]